APPLICATION OF CONTINUOUS INTEGRALS IN THE STUDY OF LASER BEAM PROPAGATION THROUGH RANDOMLY INHOMOGENEOUS MEDIA

V.M. Buldakov

Institute of Atmospheric Optics,
Siberian Branch of the Academy of Sciences of the USSR, Tomsk
Received January 14, 1991

Replacing integrating over the paths by ordinary integrating over the amplitudes of harmonics is proposed. As a result, a relation for the light field, which permits one to examine a wide class of problems for both the linear and nonlinear randomly inhomogeneous media, has been derived. Based on the proposed formula for the light field, a general relation for the spatio-temporal correlation function of arbitrary intensity fluctuations in a turbulent atmosphere has been derived.

In a linear random medium with large–scale dielectric constant inhomogeneities (in comparison with the wavelength) the propagation of an optical wave is described by the linear stochastic parabolic differential equation. It is well known that the solution of the linear parabolic equation can be represented in the form of a continuous integral.[1-4] The possibility of using it was illustrated in Refs. 5–9 by considering, by way of example, the calculation of the wave parameters for strong and weak intensity fluctuations in a turbulent atmosphere.

In order to describe the nonlinear interaction of a laser beam with random medium, the continuous integral can be used based on the notion of the T–transform.[10] The T–transform assumes an account of the time delay effect (or relaxation) for the quantities, which depend on an unknown function. In order to interpret the T–transform one can use the nonlinear differential equation, which describes the propagation of an optical wave

$$2ik\frac{\delta u(x\,\boldsymbol{p},\,t)}{\delta x}+\Delta_{\perp}u(x,\,\boldsymbol{p},\,t)+k^2\left\{\varepsilon_{\rm d}(x,\,\boldsymbol{p},\,t)\left[1-\frac{T_{\rm r}(x,\,\boldsymbol{p},\,t)+T_{\rm h}(x,\,\boldsymbol{p},\,t)}{T(x,\,\boldsymbol{p},\,t)}\right]+i\varepsilon_0(x,\,\boldsymbol{p},\,t)\right\}u(x,\,\boldsymbol{p},\,t)=0\,,\quad(1)$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength of laser beam, and x, y, and z are the Cartesian coordinates; the laser beam propagates along the x coordinate, $\boldsymbol{p} = \mathbf{i}\,y + \mathbf{j}\,z$ is the vector in a plane transverse to the direction of propagation, t is time, $u(x, y, z, t) = \varepsilon^{-ikx}\cdot E_c(x, y, z, t)$, $E_c(x, y, z, t)$ is the slowly varying complex amplitude of the wave field, $\Delta_{\perp} = \dfrac{\delta^2}{\delta y^2}+\dfrac{\delta^2}{\delta z^2}$ is the transverse Laplacian, $\varepsilon_{\rm d}(x, y, z, t)$ is the deviation of the dielectric constant of the medium from its mean value, $\varepsilon_0(x, y, z, t)$ is the imaginary part of the dielectric constant of the medium, which describes the absorption of laser beam, $T(x, y, z, t)$ is the mean temperature of the medium before its heating by the laser beam, $T_{\rm r}(x, y, z, t)$ is the deviation of temperature from $T(x, y, z, t)$ caused by the randomly inhomogeneous character of the medium, $T_{\rm h}(x, y, z, t)$ is the deviation of temperature from $T(x, y, z, t)$ caused by heating the medium by the laser beam. T_h is the functional of the function $u(x, y, z, t)$ we are interested in, i.e., $T_{\rm h}(x, y, z, t) = F[u(x, y, z, t)]$. The functional F describes the nonlinear interaction of the laser beam with the medium.

It is clear from the physical considerations that the change in the temperature of the medium $T_{\rm h}(x, y, z, t)$ occurs during time interval T. Therefore, we can consider that $T_{\rm h}(x, y, z, t)$ depends on the values of the unknown function until some preceding moment in time $t - T$. Because of the small value of T, this dependence was usually ignored. However, when constructing the difference scheme of solving the similar equations,[11] the time delay is essentially taken into account, since in order to calculate $T_{\rm h}(x, y, z, t)$ at time t_2, the values of $u(x, y, z, t)$ at the preceding time moment t_1 were used. In this respect, the notion of the T–transform introduced by V.P. Maslov[10] is closer to the real physical process than Eq. (1). As shown in Ref. 10, the method of the T–transform is closely related to the notion of nonlinear continuous integral, which can be interpreted as the method of constructing the solution by time steps. Making use of the notion of the T–transform introduced by V.P. Maslov, the nonlinear differential equation (1) can be represented in the form of a continuous integral

$$u(x,\,\boldsymbol{p},\,t)=\int\limits_{p(0)=p_0}^{p(x)=p}D\boldsymbol{p}(\xi)\,u_0(\boldsymbol{p}_0,\,t)\exp\left\{\frac{ik}{2}\int\limits_0^x{\rm d}\xi\left[\left(\frac{{\rm d}\boldsymbol{p}(\xi)}{{\rm d}\xi}\right)^2+\varepsilon_{\rm d}(\xi,\,\boldsymbol{p}(\xi),\,t)\left[1-\frac{T_{\rm r}(\xi,\,\boldsymbol{p}(\xi),\,t)+T_{\rm h}(\xi,\,\boldsymbol{p}(\xi),\,t)}{T(\xi,\,\boldsymbol{p}(\xi),\,t)}\right]+i\varepsilon_0(\xi,\,\boldsymbol{p}(\xi),\,t)\right\}\,,\quad(2)$$

where

0235-6880/91/06 439-06 $02.00

$$Dp(\xi) = \prod_{\zeta=0}^{x} \frac{dp(\xi)}{d\xi} \frac{k}{2\pi i}, \quad T_h(\xi, p(\xi), t) = F_h[u(\xi, p(\xi), t - T)] .$$

Now, there are no methods for calculating the continuous integrals with the exception of the case, in which the integral of action (in Eq. (2) this is the integral under the exponent) has a quadratic dependence. But when describing both linear and nonlinear effects the integral of action, as a rule, cannot be represented in such a form. In this connection, a necessity of employing the approximate methods for the calculations arises. In particular, the approximations taking into account solely a single path, which makes the main contribution to the continuous integral, can be developed. In Ref. 12, it was proposed to choose the straight lines, which join two end points, as such a path. In order to determine the main path, one can employ Euler's equation which has a solution under conditions of essential approximations and limitations.[13]

In this paper, we propose to take into account any paths with the help of their representation in the form of a superposition of a certain arbitrary oriented straight line and a collection of different harmonics. In so doing, integrating over the paths is replaced by ordinary integration over the amplitudes of harmonics.

It is well known[2,10] that integration over the paths in Eq. (2) is analogous to taking the limit

$$u(x, p, t) = \lim_{N\to\infty}\left(\frac{k}{2\pi i\Delta x}\right)^N \int_{-\infty}^{\infty}\cdots\int d^2\rho_0\, d^2\rho_1 \ldots d^2\rho_{N-1}\, u_0(p_0, t) \exp\left\{\frac{ik}{2}\sum_{j=1}^{N}\left[\frac{(p_j - p_{j-1})^2}{(\Delta x)^2} + \right.\right.$$

$$\left.\left. + \varepsilon_d(j\Delta x, p_j, t)\left[1 - \frac{T_r(j\Delta x, p_j, t) + T_h(j\Delta x, p_j, t)}{T(j\Delta x, p_j, t)}\right] + i\varepsilon_0(j\Delta x, p_j, t)\right]\Delta x\right\}, \quad (3)$$

where $p_N = p$ and $\Delta x = x/N$.

Formulas (2) and (3) mean that $u(x, p, t)$ is determined as a sum (integral) over the paths, which arbitrarily join all points of the source to the point **p** located at the end of the path. The contributions of different paths to the sum can be arbitrary including the case, in which they are opposite to each other. In order to perform a convenient analysis of the contribution of different paths to the sum, let us represent the path in the following way:

$$p(\xi) = p_s(\xi) + p_d(\xi) , \quad (4)$$

where $p_s(\xi) = (1 - \xi/x)\, p_0 + \xi/x\, p$ is the straight line, which joins the end points p_0 and p, $p_d(\xi)$ is the path, which has the property $p_d(0) = p_d(x) = 0$. According to Steklov's theorem,[14] the path $p_d(\chi)$ can be expanded into a series

$$p_d(\xi) = \sum_{l=1}^{\infty} \mathbf{a}_l\, \varphi_l(\xi) , \quad (5)$$

where the functions $\varphi_l(\xi) = \sqrt{2/x}\,\sin(l\pi\xi/x)$ are used as an orthonormal system of functions. The values $\mathbf{a}_l = \int_0^x d\xi p_d(\xi)\varphi_l(\xi)$ are unknown because of the arbitrary shape of the path $p_d(\xi)$. If we represent Eq. (5) in the form

$$p_d(\xi) = \sum_{l=1}^{N-1} \mathbf{a}_l\, \varphi_l(\xi) + \sum_{l=N}^{\infty} \mathbf{a}_l\, \varphi_l(\xi) \quad (5')$$

and neglect the last sum, whose contribution decreases as N increases, we can convert from the variables of integration p_j in Eq. (3) to new variables of integration a_l

$$p_j = \left(1 - \frac{j}{N}\right) p_0 + \frac{j}{N}\, p + \sum_{l=1}^{N-1} \mathbf{a}_l\, \varphi_l\left(\frac{j}{N}\, x\right) .$$

In addition, in order to obtain a convenient representation of Eq. (3), we carry out the following change of variables:

$$\mathbf{a}_l = \frac{\sqrt{x}}{2N\sin(e\pi/2N)}\, \mathbf{b}_l .$$

In this case, the Jacobian of the transform from the variables of integrating p_j to b_l is N^{-N}, while the summation over j can be replaced by integration over the path. Hence, the expression for the light field assumes the form

$$u(x, p, t) = \int_{\infty}^{-\infty} d^2\rho_0\, u_0(p_0, t)\, \frac{k}{2\pi ix} \lim_{N\to\infty}\left(\frac{k}{2\pi ix}\right)^{N-1} \int_{-\infty}^{\infty}\cdots\int d^2b_1 \ldots d^2b_{N-1} \exp\left\{\frac{ik}{2x}\left[(p - p_0)^2 + \sum_{l=1}^{N-1} b_l^2\right] + \right.$$

$$+ \frac{ik}{2}\int_0^x dx' \left[\varepsilon_d\left(x', \left(1 - \frac{x'}{x}\right)p_0 + \frac{x'}{x}\, p + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{b}_l, t\right)\left[1 - \frac{T_r\left(x', \left(1 - \frac{x'}{x}\right)p_0 + \frac{x'}{x}\, p + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{b}_l, t\right)}{T\left(x', \left(1 - \frac{x'}{x}\right)p_0 + \frac{x'}{x}\, p + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{b}_l, t\right)} - \right.\right.$$

$$-\frac{T_{\mathrm{h}}\left(x', \left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum\limits_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)}{T\left(x', \left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum\limits_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)}\Bigg]+i\varepsilon_0\left(x', \left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)\Bigg]\Bigg\}, \tag{6}$$

where $\nu_i(x') = \sin(l\pi x'/x)/\sqrt{2N}\,\sin(l\pi/2N))$.

Since the scales of variation of the values ε_d, T and ε_0, as a rule, exceed the dimensions of the laser beams, and ε_d, T and ε_0 are stationary within the time over which the pulse acts, the dependence of these values on the transverse coordinates and time can be ignored. In this case,

$$u(x,\boldsymbol{p},t)=\frac{k}{2\pi ix}\exp\left\{-\frac{1}{2}\int_0^x dx'\alpha(x')+\frac{ik}{2}\int_0^x dx'\varepsilon_d(x')\right\}\int_{-\infty}^{\infty} d^2\boldsymbol{p}_0 u_0(\boldsymbol{p}_0,t)\exp\left\{\frac{ik}{2x}(\boldsymbol{p}-\boldsymbol{p}_0)^2\right\}\lim_{N\to\infty}\left(\frac{k}{2\pi ix}\right)^{N-1}\times$$

$$\times\int_{-\infty}^{\infty}\ldots\int d^2b_1\ldots d^2b_{N-1}\exp\left\{\frac{ik}{2x}\sum_{l=1}^{N-1}b_l^2+\frac{ik}{2}\int_0^x dx'\varepsilon_2(x')\left[T_{\mathrm{r}}\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)+\right.\right.$$

$$\left.\left.+T_{\mathrm{h}}\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)\right]\right\}, \tag{7}$$

where $\alpha(x') = k\varepsilon_0(x')$ is the molecular absorption coefficient and $\varepsilon_2(x') = -\varepsilon_d(x')/T(x')$. In the wavelength range $\lambda = 0.2 - 20$ μm we can use the following approximate formula[15]: $\varepsilon_d(x') = 2\cdot10^{-6}\dfrac{P(x')}{I(x')}(77.6 + 0.584\lambda^{-2})$. Under standard conditions $P(x') = 1013.25$ mbar, $T(x') = 288$ K,

$$\varepsilon_2=\begin{cases}-2.25\cdot10^{-6}\,\frac{1}{\mathrm{K}}\ \text{for}\ \lambda=0.2\ \text{mm}\\ -1.89\cdot10^{-6}\,\frac{1}{\mathrm{K}}\ \text{for}\ \lambda=20\ \text{mm}\end{cases}$$

Usually ε_2 is assumed to be equal to $-2\cdot10^{-6}$ 1/K. The value

$$\varepsilon_2(x')T_{\mathrm{r}}\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)=\varepsilon_1T_{\mathrm{r}}\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p}+\sum_{l=1}^{N-1}\nu_l(x')\mathbf{b}_l,\ t\right)$$

describes the dielectric constant fluctuations in the medium due to its random inhomogeneity. If we neglect in Eq. (7) the integration over the amplitudes of the harmonics b_l then we obtain the expression

$$u(x,\boldsymbol{p},t)=\frac{k}{2\pi ix}\exp\left\{-\frac{1}{2}\int_0^x dx'\alpha(x')+\frac{ik}{2}\int_0^x dx'\varepsilon_d(x')\right\}\int_{-\infty}^{\infty} d^2\boldsymbol{p}_0\,u_0(\boldsymbol{p}_0,t)\exp\left\{\frac{ik}{2x}(\boldsymbol{p}-\boldsymbol{p}_0)^2+\frac{ik}{2}\int_0^x dx'\right.\times$$

$$\left.\times\left[\varepsilon_1\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p},t\right)+\varepsilon_2(x')\,T_{\mathrm{h}}\left(x',\left(1-\frac{x'}{x}\right)\boldsymbol{p}_0+\frac{x'}{x}\boldsymbol{p},\ t\right)\right]\right\}, \tag{8}$$

which agrees with the previously well–known formula of the light field in the phase approximation of the Huygens–Kirchhoff method.[16] It is clear from the physical considerations that the first low–frequency harmonics make the main contribution to integral (7). For this reason, on the one hand, we can use a small number of harmonics for the calculations and, on the other, we can determine the accuracy of the accepted restrictions.

Formula (7) for the field of laser beam in the case of linear ($T_{\mathrm{h}} = 0$) and nonlinear interactions with both stationary and moving randomly inhomogeneous media makes it possible to investigate a wide class of problems for the one–way and round–trip paths.

A possibility of employing Eq. (7) can be demonstrated by considering, by way of example, a derivation of relations for the spatio–temporal correlation function of the intensity fluctuations B_I in a turbulent atmosphere for $T_{\mathrm{h}} = 0$

$$B_I(x,\boldsymbol{p}_1,\boldsymbol{p}_2,\tau)=\langle I(x,\boldsymbol{p}_1,t)\rangle-\langle I(x,\boldsymbol{p}_2,t+\tau)\rangle-$$

$$-\langle I(x,\boldsymbol{p}_1,t)\rangle\langle I(x,\boldsymbol{p}_2,t+\tau)\rangle\ , \tag{9}$$

where $I(x, \boldsymbol{p}, t) = u(x, \boldsymbol{p}, t)u^*(x, \boldsymbol{p}, t)$ is the intensity of laser beam and the angular brackets <...> denote the statistical averaging over the ensemble of realizations of the dielectric constant of the medium ε_1. Hereafter, we shall use the Gaussian distribution law and the assumption of δ–correlate fluctuations ε_1 as well as condition that the turbulence be "frozen".[17] In this case, with account of Eq. (7), we can write down the following general relation

$$<I(x, \boldsymbol{p}, t)> = \int_{-\infty}^{\infty} d^4\rho_0 R <u_0\left(\mathbf{R} + \frac{\boldsymbol{p}_0}{2}, t\right) u_0^*\left(\mathbf{R} - \frac{\boldsymbol{p}_0}{2}, t\right)>_s <G\left(0, \mathbf{R} + \frac{\boldsymbol{p}_0}{2}; x, \boldsymbol{p}, t\right) G^*\left(0, \mathbf{R} - \frac{\boldsymbol{p}_0}{2}; x, \boldsymbol{p}, t\right)>; \quad (10)$$

$$<I(x, \boldsymbol{p}_1, t)\, I(x, \boldsymbol{p}_2, t+\tau)> = \int_{-\infty}^{\infty} d^8R_1, R_2, R_3, R_4 <u_0\left(\frac{1}{2}(R_1 + R_2 + R_3 + R_4), t\right) \times$$
$$\times u_0^*\left(\frac{1}{2}(R_1 + R_2 - R_3 - R_4), t\right) u_0\left(\frac{1}{2}(R_1 - R_2 - R_3 + R_4), t\right) u_0^*\left(\frac{1}{2}(R_1 - R_2 + R_3 - R_4), t\right)>_s \times$$
$$\times <G\left(0, \frac{1}{2}(R_1 + R_2 + R_3 + R_4); x, \boldsymbol{p}_1, t\right) G^*\left(0, \frac{1}{2}(R_1 + R_2 - R_3 - R_4); x, \boldsymbol{p}_1, t\right) \times$$
$$\times G\left(0, \frac{1}{2}(R_1 - R_2 - R_3 + R_4); x, \boldsymbol{p}_2, t+\tau\right) G^*\left(0, \frac{1}{2}(R_1 - R_2 + R_3 - R_4); x, \boldsymbol{p}_2, t+\tau\right)>. \quad (11)$$

The angular brackets $<...>_s$ denote averaging over the source fluctuations, $G(0, \mathbf{R}, x, \boldsymbol{p}, t)$ is the Green's function,

$$<G\left(0, \mathbf{R} + \frac{\boldsymbol{p}_0}{2}; x, \boldsymbol{p}, t\right) G^*\left(0, \mathbf{R} - \frac{\boldsymbol{p}_0}{2}; x, \boldsymbol{p}, t\right)> = \left(\frac{k}{2\pi x}\right)^2 \exp\left\{\frac{ik}{x}\boldsymbol{p}_0(\mathbf{R} - \boldsymbol{p}) - \frac{\pi k^2}{4}\int_0^x dx'\, H\left(\left(1 - \frac{x'}{x}\right)\boldsymbol{p}_0\right)\right\} \quad (12)$$

$$<G\left(0, \frac{1}{2}(R_1 + R_2 + R_3 + R_4); x, \boldsymbol{p}_1, t\right) G^*\left(0, \frac{1}{2}(R_1 + R_2 - R_3 - R_4); x, \boldsymbol{p}_1, t\right) \times$$
$$\times G\left(0, \frac{1}{2}(R_1 - R_2 - R_3 + R_4); x, \boldsymbol{p}_2, t+\tau\right) G^*\left(0, \frac{1}{2}(R_1 - R_2 + R_3 - R_4); x, \boldsymbol{p}_2, t+\tau\right)> =$$
$$= \left(\frac{k}{2\pi x}\right)^4 \exp\left\{\frac{ik}{x}[\mathbf{R}_1\mathbf{R}_4 + \mathbf{R}_2\mathbf{R}_3 - \boldsymbol{p}_1(\mathbf{R}_3 + \mathbf{R}_4) + \boldsymbol{p}_2(\mathbf{R}_3 - \mathbf{R}_4)]\right\} f(\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4, x, \boldsymbol{p}_1, \boldsymbol{p}_2, \tau); \quad (13)$$

$$H(\boldsymbol{p}) = 2\int_{-\infty}^{\infty} d^2\mathbf{k}\ \Phi_\varepsilon(\mathbf{k})\,[1 - e^{\pm i\kappa \boldsymbol{p}}]; \quad (14)$$

where $\Phi_\varepsilon(\kappa)$ is the three–dimensional spatial spectrum of the dielectric constant fluctuations;

$$f(\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4, x, \boldsymbol{p}_1, \boldsymbol{p}_2, \tau) = \lim_{N\to\infty}\left(\frac{k}{2\pi i x}\right)^{2(N-1)} \int_{-\infty}^{\infty} d^{2(N-1)}b_1, \ldots, b_{N-1}, B_1, \ldots, B_{N-1} \exp\left\{\frac{ik}{x}\sum_{l=1}^{N-1} b_l\mathbf{B}_l\right\} \times$$
$$\times \exp\left\{-\frac{\pi k^2}{4}\int_0^x dx' \left[H\left(\left(1 - \frac{x'}{x}\right)(\mathbf{R}_3 + \mathbf{R}_4) + \sum_{l=1}^{N-1}\nu_l(x')\mathbf{B}_l\right) -\right.\right.$$
$$- H\left(\left(1 - \frac{x'}{x}\right)(\mathbf{R}_2 + \mathbf{R}_3) + \frac{x}{x'}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \sum_{l=1}^{N-1}\nu_l(x')(\mathbf{b}_l + \mathbf{B}_l) + \mathbf{v}(x')\,\tau\right) + H\left(\left(1 - \frac{x'}{x}\right)(\mathbf{R}_2 + \mathbf{R}_4) +\right.$$

$$+ \frac{x}{x'}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{b}_l + \mathbf{v}(x')\,\tau\Big) + H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 - \mathbf{R}_4) + \frac{x}{x'}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{b}_l + \mathbf{v}(x')\,\tau\Big) -$$

$$- H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 - \mathbf{R}_3) + \frac{x}{x'}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \sum_{l=1}^{N-1} \nu_l(x')(\mathbf{b}_l - \mathbf{B}_l) + \mathbf{v}(x')\,\tau\Big) +$$

$$+ H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_3 - \mathbf{R}_4) + \sum_{l=1}^{N-1} \nu_l(x')\mathbf{B}_l\Big)\Big]\Big\}; \qquad (15)$$

$\mathbf{v}(x')$ is the vector of wind velocity. As a rule, the analyses of the fluctuation characteristics given by Eq. (8) for different values of $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and τ are performed for weak ($\beta_0^2 < 1$) and strong ($\beta_0^2 \gg 1$) intensity fluctuations. Here, $\beta_0^2 = 0.31 C_\varepsilon^2 k^{7/6} x^{11/6}$ is the effective parameter, which determines the intensity of turbulence on the path. C_ε^2 is the structural parameter of the dielectric constant fluctuations. Since fluctuation H is proportional to the parameter β_0^2, the second exponent in Eq. (15) for $\beta_0^2 < 1$ can be represented in the form $e^x \approx 1 + x$. In this case, we succeed in integrating over the variables $\mathbf{b}_l$ and $\mathbf{B}_l$ and in determining the limiting value

$$f(\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4, x, \boldsymbol{p}_1, \boldsymbol{p}_2, \tau) \approx 1 - \frac{\pi k^2}{2}\int_0^x \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}^2\mathbf{k}\,\Phi_\varepsilon(\mathbf{k})\Big[2 - \exp\Big\{i\Big(1 - \frac{x'}{x}\Big)\mathbf{k}(\mathbf{R}_3 + \mathbf{R}_4)\Big\} - \exp\Big\{i\Big(1 - \frac{x'}{x}\Big)\mathbf{k}(\mathbf{R}_3 - \mathbf{R}_4)\Big\}\Big] -$$

$$- \exp\Big\{i\mathbf{k}\Big[\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big]\Big\} - \exp\Big\{i\mathbf{k}\Big[\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 - \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big]\Big\} +$$

$$+ \exp\Big\{-i\frac{x'}{k}\Big(1 - \frac{x'}{x}\Big)\mathbf{k}^2 + i\mathbf{k}\Big[\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_3) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big]\Big\} + \exp\Big\{i\frac{x'}{k}\Big(1 - \frac{x'}{x}\Big)\mathbf{k}^2 +$$

$$+ i\mathbf{k}\Big[\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_3) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big]. \qquad (16)$$

For strong intensity fluctuations ($\beta_0^2 \gg 1$), one can use an asymptotic expansion described in Refs. 5 and 6 which allows one to carry out the integration over $\mathbf{b}_l$ and $\mathbf{B}_l$ in Eq. (15) and to find the limit

$$f(\mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4, x, \boldsymbol{p}_1, \boldsymbol{p}_2, \tau) \approx \exp\Big\{-\frac{\pi k^2}{4}\int_0^x \mathrm{d}x'\Big[H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big) +$$

$$+ H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 - \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) + \mathbf{v}(x')\,\tau\Big)\Big]\Big\} + \pi k^2 \int_0^x \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}^2\mathbf{k}\,\Phi_\varepsilon(x'', \mathbf{k})\exp\Big\{i\mathbf{k}\Big(1 - \frac{x''}{x}\Big)\mathbf{R}_3\Big\}\times$$

$$\times\Big[\cos\Big(\Big(1 - \frac{x''}{x}\Big)\mathbf{k}\mathbf{R}_4\Big) - \cos\Big[\mathbf{k}\Big[\Big(1 - \frac{x''}{x}\Big)\mathbf{R}_2 + \frac{x''}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) - \frac{x''}{k}\Big(1 - \frac{x''}{x}\Big)\mathbf{k} + \mathbf{v}(x'')\,\tau\Big]\Big]\Big]\times$$

$$\times \exp\Big\{-\frac{\pi k^2}{4}\int_0^x \mathrm{d}x'\Big[H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) - \frac{x'}{k}\Big(1 - \frac{x''}{x}\Big)\mathbf{k} + \mathbf{v}(x')\,\tau\Big) + H\Big(\Big(1 - \frac{x'}{x}\Big)\times$$

$$\times(\mathbf{R}_2 - \mathbf{R}_4) + \frac{x'}{x}(\boldsymbol{p}_1 - \boldsymbol{p}_2) - \frac{x'}{k}\Big(1 - \frac{x''}{x}\Big)\mathbf{k} + \mathbf{v}(x')\,\tau\Big)\Big]\int_{x''}^x \mathrm{d}x'\Big[H\Big(\Big(1 - \frac{x'}{x}\Big)(\mathbf{R}_2 + \mathbf{R}_4) + \frac{x'}{x} + (\boldsymbol{p}_1 - \boldsymbol{p}_2) -$$

$$-\frac{x''}{k}\left(1-\frac{x''}{x}\right)\mathbf{k}+\mathbf{v}(x')\,\tau\Big)+H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_2-\mathbf{R}_4)+\frac{x'}{x}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)-\frac{x''}{k}\left(1-\frac{x'}{x}\right)\mathbf{k}+\mathbf{v}(x')\,\tau\right)\Bigg]\Bigg]\Bigg\}+$$

$$+\exp\left\{-\frac{\pi k^2}{4}\int\limits_0^x\mathrm{d}x'\left[H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3+\mathbf{R}_4)\right)+H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3-\mathbf{R}_4)\right)\right]\right\}+\pi k^2\int\limits_0^x\mathrm{d}x''\int\limits_{-\infty}^{\infty}\mathrm{d}^2\mathbf{k}\,\Phi_\varepsilon(x'',\mathbf{k})\times$$

$$\times\exp\left\{i\mathbf{k}\left[\left(1-\frac{x''}{x}\right)\mathbf{R}_2+\frac{x''}{x}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)+\mathbf{v}(x'')\,\tau\right]\right\}\left[\cos\left(\left(1-\frac{x''}{x}\right)\mathbf{k}\mathbf{R}_4\right)-\cos\left(\mathbf{k}\left[\left(1-\frac{x''}{x}\right)\mathbf{R}_3-\frac{x''}{k}\left(1-\frac{x''}{k}\right)\mathbf{k}\right]\right)\right]\times$$

$$\times\exp\left\{-\frac{\pi k^2}{4}\int\limits_0^x\mathrm{d}x'\left[H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3+\mathbf{R}_4)-\frac{x'}{k}\left(1-\frac{x''}{x}\right)\mathbf{k}\right)+H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3-\mathbf{R}_4)-\frac{x'}{k}\left(1-\frac{x''}{x}\right)\mathbf{k}\right)\right]+\right.$$

$$\left.+\int\limits_{x''}^x\mathrm{d}x'\left[H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3+\mathbf{R}_4)-\frac{x''}{k}\left(1-\frac{x'}{x}\right)\mathbf{k}\right)+H\left(\left(1-\frac{x'}{x}\right)(\mathbf{R}_3-\mathbf{R}_4)-\frac{x''}{k}\left(1-\frac{x'}{x}\right)\mathbf{k}\right)\right]\right]\right\}. \quad (17)$$

The phase approximation of the Huygens–Kirchhoff method does not describe the intensity fluctuations of the spherical wave and of an incoherent source. For the above–derived relations, the similar limitations are absent. As an example, we can obtain the relative variance of the intensity fluctuations of the spherical wave $\sigma_I^2 = B_I(x, 0, 0, 0)/\langle I((x, 0, t)\rangle^2$. For calculations, let us use the Kolmogorov spectrum of the dielectric constant fluctuations, namely, $\Phi_\varepsilon(\kappa) = 0.033C_\varepsilon^2 x^{-11/3}$. For the spherical wave $u_0(\boldsymbol{\rho}, t) = u_0\frac{2\pi}{k^2}\sigma(\boldsymbol{\rho})$, and $\langle I(x, 0, t)\rangle = (u_0/kx)^2$. Using Eqs. (9), (11), (13), and (16) for weak intensity fluctuations ($\beta_0^2 < 1$) we obtain

$$\sigma_I^2 = 0.4\beta_0^2 + 0(\beta_0^4)\,, \quad (18)$$

and for strong intensity fluctuations ($\beta_0^2 \gg 1$) (formulas (9), (11), (13), and (17) are used), we obtain

$$\sigma_I^2 = 1 + 2.73\beta_0^{-4/5} + 0(\beta_0^{-8/5})\,. \quad (19)$$

The derived formulas (18) and (19) coincide with the previously well–known results.[6,17]

REFERENCES

1. I.M. Gel'fand and A.M. Yaglom, Usp. Matem. Nauk **2**, No. 1 (67), 77–114 (1956).
2. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw–Hill, N. Y., 1965).
3. V.I. Klyatskin, *Statistical Description of Dynamic Systems with Fluctuating Parameters* (Nauka, Moscow, 1975), 239 p.
4. V.I. Tatarskii, in: *Collection of Scientific Papers. Monte Carlo Methods in Computing Mathematics and Mathematical Physics*, Computing Center of the Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk (1976), pp. 75–90.
5. V.U. Zavorotnyi, V.I. Klyatskin, and V.I. Tatarskii, Zh. Eksp. Teor. Fiz. **73**, No. 2 (8), 481–497 (1977).
6. M.I. Charnotskii, in: *Abstracts of Reports at the 5th All–Union Symposium on Propagation of Laser Radiation in the Atmosphere*, Tomsk (1979), Vol. 2, pp. 74–78.
7. V.I. Tatarskii and V.U. Zavorotnyi, Proc. Soc. of Photo–Opt. Instrum. Eng. **642**, 276–281 (1986).
8. P.A. Bakut, in: *Abstracts of Reports at the 10th All–Union Symposium on Propagation of Laser Radiation in the Atmosphere*, Tomsk (1989), pp. 17–20.
9. R.Kh. Almaev and A.A. Suvorov, Pis'ma Zh. Eksp. Teor. Fiz. **52**, No. 2, 718–722, (1990).
10. V.P. Maslov, *Complex Markov's Chains and Continuous Feynman's Integrals (for Nonlinear Equations)* (Nauka, Moscow, 1976), 192 p.
11. A.A. Samarskii, *Introduction to the Theory of Difference Schemes* (Nauka, Moscow, 1971), 552 p.
12. A.S. Chirkin and F.M. Yusubov, Kvant. Elektron. **10**, No. 9, 1833–1842 (1983).
13. A.S. Chirkin and F.M. Yusubov, Vestn. Mosk. Univ. Ser. Fiz. Astron. **26**, No. 4, 57–62 (1985).
14. A.N. Tikhonov and A.A. Samarskii, *Equations of Mathematical Physics* (Pergamon Press, Oxford, 1964).
15. *Reference Book on Geophysics* (Nauka, Moscow, 1965).
16. V.A. Banakh, V.M. Buldakov, and V.L. Mironov, Kvantovaya Elektron. **13**, No. 6, 1220–1226 (1986).
17. V.I. Tatarskii, *Wave Propagation in a Turbulent Medium* (McGraw–Hill, New York, 1961).