ON THE SOLUTION OF LIDAR EQUATION OF TOMOGRAPHIC SOUNDING OF THE ATMOSPHERE

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The problem of reconstructing the two–dimensional distributions of the attenuation and backscattering coefficients from the data of the airborne tomographic sounding of the atmosphere is considered. Solutions of the integral equations of two–beam and three–beam tomographic lidar sounding have been obtained analytically. It has been shown that in this case it is not unnecessary to use any a priori *assumption regarding the characteristics sought for to reconstruct the attenuation and backscattering coefficients. The solutions obtained in this paper can serve as the basis for the construction of new calculational algorithms for tomographic laser sounding of the atmosphere.*

At present the number of investigations associated with the airborne lidar study of the spatial distribution of aerosol and gaseous components of the atmosphere increases.[1-3] To solve such problems under conditions in which in the process of motion of the lidar relative to the region of space under study sounding is performed along different directions, the new promising methods of interpretation of lidar data may be proposed based on the ideas of reconstructive tomography.[4] Such an approach was developed in Refs. 5 and 6, where the interrelation between the methods of transmission tomography and lidar sounding was obtained and the numerical algorithms for interpreting the data of two–beam and three-beam tomographic sounding of the atmosphere from airborne lidar were presented. In this paper the solutions of the integral equations of two-beam and three–beam tomographic laser sounding of the atmosphere are obtained analytically which may serve as a basis for construction of new computational algorithms.

1. Statement of the problem. We shall consider the mathematical formulation of the problem of tomographic sounding of the atmosphere from airborne lidar. An analysis is performed for a two-dimensional observation scheme similar to that considered in Ref. 6 and is shown in Fig. 1. The plane of the figure coincides with the sounding plane passing through the direction of lidar motion and the direction of sounding. We fix the Cartesian coordinates (x, z) on the sounding plane with the x axis coinciding with the direction of lidar motion. For definiteness we assume that the lidar moves along the straight line in the horizontal plane, while sounding is carried out in the direction $\mathbf{n} = (\sin\varpi, \cos\varpi)$ characterized by the polar angle ϖ measured from the direction toward the nadir. Let us assume that the lidar is located at the point $\mathbf{r}^* = (x^*, z^*)$. Then the lidar signal from scattering volume located at the point $\mathbf{r} = (x, z)$ in the single scattering approximation is given by the relation

$$S(\mathbf{r}^*, \rho, \mathbf{n}) = \beta(\mathbf{r}^* + \rho\mathbf{n})\exp\left\{-2\int\limits_0^\rho a(\mathbf{r}^* + \rho'\mathbf{n})d\rho'\right\}, \qquad (1)$$

where ρ is the distance between the lidar and the scattering volume, $\mathbf{r} = \mathbf{r}^* + \rho\mathbf{n}$, $S(\mathbf{r}^*, \rho, \mathbf{n}) = P(\mathbf{r}^*, \rho, \mathbf{n})\rho^2/P_0A$, where P_0 and $P(\mathbf{r}^*, \rho, \mathbf{n})$ are the power of transmitted and received signals, respectively, A is the instrumental constant, $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ are the attenuation and backscattering coefficients at the point $\mathbf{r}$. Moving the lidar along the x–axis we perform sounding in different directions $\mathbf{n}$. The problem is to reconstruct the spatial distribution of the optical characteristics $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ of the atmosphere from the received lidar signals. It was shown in Ref. 6 that, in contrast to the transmission tomography, it is sufficient to perform sounding along two directions to solve the formulated problem. In addition, one of the numerical algorithm available for tomographic processing of the lidar signals was considered.

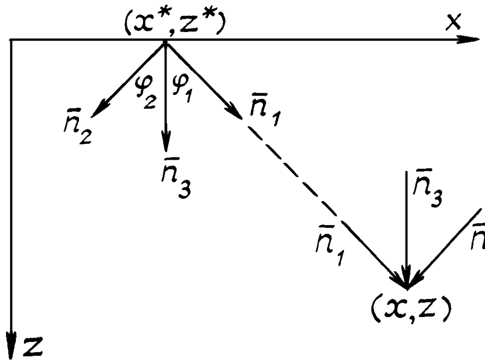

FIG. 1.

To construct the analytical solution of Eq. (1) we transfer from the initial equation to the differential one. For this purpose we take the logarithm of the left and right sides of Eq. (1) and then differentiate both sides with respect to the direction $\mathbf{n}$. As a result we obtain

$$\frac{\partial \ln S(\mathbf{r}^*, \rho, \mathbf{n})}{\partial \mathbf{n}} = \frac{\partial \ln\beta(\mathbf{r})}{\partial \mathbf{n}} - 2\alpha(\mathbf{r}) . \qquad (2)$$

Taking into account the fact that the derivative with respect to the direction $\mathbf{n}$ of the arbitrary differential function $f(x, z)$ fixed on the plane has the form

$$\frac{\partial f(x, z)}{\partial \mathbf{n}} = \frac{\partial f}{\partial x}\sin\varphi + \frac{\partial f}{\partial z}\cos\varphi ,$$

we finally have

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$$\frac{\partial G(\mathbf{r}^*, \rho, \mathbf{n})}{\partial \mathbf{n}} == \frac{\partial L(x, z)}{\partial \mathbf{x}} \sin\varphi + \frac{\partial L(x, z)}{\partial \mathbf{z}} \cos\varphi - 2\alpha(x, z) , \quad (3)$$

where

$$G(\mathbf{r}^*, \rho, \mathbf{n}) = \ln S(\mathbf{r}^*, \rho, \mathbf{n}) ; \quad L(x, z) = \ln\beta(\mathbf{r}) .$$

Equation (3) is the basic equation of lidar tomographic sounding in the differential form. Given that the value of the polar angle φ is fixed, Eq. (3) represents the first–order partial differential equation in the backscattering coefficient $\beta(x, z)$ which can be solved by standard methods. In addition, the salient feature of Eq. (3) is that it includes one more unknown function, namely, the attenuation coefficient $\alpha(\mathbf{r})$. Therefore, to solve Eq. (3) for the functions $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ we should start from the fact that Eq. (3) describes the family of equations depending parametrically on the polar angle φ. In this case, the variety and complexity of the computational systems of tomographic processing of the lidar signals will depend on the number and the magnitude of the polar angles φ_i. In the simplest case it is sufficient to fix two polar angles $\varphi_i (i = 1, 2)$.

2. Two–beam scheme. In the considered case each point $\mathbf{r} = (x, z)$ of the region under study is sounded from two different directions $\mathbf{n}_1$ and $\mathbf{n}_2$ which are characterized by the zenith angles φ_1 and φ_2 when the lidar is located at the points $\overset{*}{\mathbf{r}}_i = \mathbf{r} - \rho\mathbf{n}_i$ $(i = 1, 2)$. The corresponding system of equations has the form

$$\frac{\partial L(x, z)}{\partial x} \sin\varphi_i + \frac{\partial L(x, z)}{\partial z} \cos\varphi_i - 2\alpha(x, z) = \frac{\partial G_i}{\partial \mathbf{n}_i} , \quad (4)$$

where $G_i = G(\overset{*}{\mathbf{r}}_i, \rho, \mathbf{n})$ and $i = 1, 2$.

Eliminating the unknown function $\alpha(x, z)$ among the system of Eqs. (4) we can obtain the partial differential equation for the function $L(x, z)$

$$a \frac{\partial L(x, z)}{\partial x} + b \frac{\partial L(x, z)}{\partial z} = f(x, z) \quad (5)$$

with the right side

$$f(x, z) = \frac{\partial G_1}{\partial \mathbf{n}_1} - \frac{\partial G_2}{\partial \mathbf{n}_2} \quad (6)$$

and with the coefficients

$$a = \sin\varphi_1 - \sin\varphi_2 ; \quad b = \cos\varphi_1 - \cos\varphi_2 . \quad (7)$$

The boundary conditions are required to obtain the unique solution of Eq. (5). It is most natural to specify the function $L(x, z)$ at the upper boundary of the sounded region $(z = 0)$:

$$(L(x, z = 0) = L_0(x) . \quad (8)$$

In so doing the function $L_0(x)$ can be easily determined experimentally since it is related uniquely to the magnitude of lidar signal from the upper edge of the region $L_0 = G(\overset{*}{\mathbf{r}}_i, 0, \mathbf{n})$. The solution of Eq. (5) by the method of characteristics with boundary condition (8) have the form

$$L(x, z) = L_0\left(x - \frac{a}{b} z\right) + \frac{1}{b} \int\limits_0^z f\left(x - \frac{a}{b} \zeta, \ z - \zeta\right) d\zeta . \quad (9)$$

or going over to the backscattering coefficient

$$\beta(x, z) = \beta_0\left(x - \frac{a}{b} z\right) \exp\left\{\frac{1}{b} \int\limits_0^z f\left(x - \frac{a}{b} \zeta, \ z - \zeta\right) d\zeta\right\} . \quad (10)$$

Formulas (9) and (10) are valid under condition when $\cos \varphi_1 \neq \cos \varphi_2$. The integrand $f(x, z)$ in formulas (9) and (10) is given by the relation

$$f(x, z) \frac{\partial}{\partial x} \left[G_1\sin\varphi_1 - G_2\sin\varphi_2 \right] + \frac{\partial}{\partial z} \left[G_1\cos\varphi_1 - G_2\cos\varphi_2 \right], (11)$$

in which the arguments x and z are substituted by

$$p = x - \frac{a}{b} \zeta; \quad q = z - \xi, \quad (12)$$

while the notations are preserved for the functions $G_1 = G_1(x, z)$ and $G_2 = G_2(x, z)$ depending on the Cartesian coordinates. It should be noted that the partial derivatives of functions $G_i(p, q)$ $(i = 1, 2)$ are related by the equation

$$\frac{\partial G_i}{\partial p} = - \frac{b}{a} \left(\frac{\partial G_i}{\partial z} + \frac{\partial G_i}{\partial q} \right) , \quad i = 1, 2. \quad (13)$$

By substituting Eq. (11) into formula (9) on account of Eqs. (12) and (13) one may show that

$$L(x, z) = k_1 G_1(x, z) - k_2 G_2(x, z) - c \int\limits_0^z \frac{\partial \Omega(p, q)}{\partial q} d\zeta, \quad (14)$$

where

$$k_1 = \frac{\sin\varphi_1}{\sin\varphi_1 - \sin\varphi_2} , \quad k_2 = \frac{\sin\varphi_2}{\sin\varphi_1 - \sin\varphi_2} ,$$

$$c = \mathrm{cotan}\left(\frac{\varphi_1 - \varphi_2}{2}\right) \Big/ \sin(\varphi_1 + \varphi_2) ,$$

$$\Omega(p, q) = \left(G_1(p, q) - G_2(p, q)\right) .$$

It follows from Eq. (14) that the final equation solving the problem of two–beam tomographic sounding of the atmosphere for backscattering coefficient has the form

$$\beta(x, z) = \frac{S_1^{k_1}(x, z)}{S_2^{k_2}(x, z)} \exp\left\{-c \int\limits_0^z \frac{\partial \Omega(p, q)}{\partial q} d\zeta\right\} . \quad (15)$$

By substituting the function $\beta(x, z)$ from Eq. (15) into the first equation of system (4) one may obtain the solution for the attenuation coefficient after simple manipulation

$$\alpha(x, z) = \left\{ \nu \frac{\partial \Omega(x, z)}{\partial x} + \eta \frac{\partial \Omega(x, z)}{\partial z} + \tau(x, z) \right\} \Big/ 2 , \quad (16)$$

where

$$\nu = \frac{\sin\varphi_1 \sin\varphi_2}{\sin\varphi_1 - \sin\varphi_2}, \quad \eta = \frac{\cos\varphi_1 \cos\varphi_2}{\cos\varphi_1 - \cos\varphi_2},$$

$$\tau(x, z) = \frac{1 + \cos(\varphi_1 - \varphi_2)}{b\sin(\varphi_1 + \varphi_2)} \int_0^z \frac{\partial^2 \Omega(p, q)}{\partial p\, \partial q} \mathrm{d}\zeta,$$

while the function $\Omega(x, z) = \ln[S_1(x, z)/S_2(x, z)]$ is determined from the ratio of lidar signals coming from the point $\mathbf{r} = (x, z)$ from two different directions $\mathbf{n}_1$ and $\mathbf{n}_2$.

3. Symmetrical two–beam scheme. As it was mentioned above, solution (15) for $\beta(x, z)$ was obtained under condition $\cos\varphi_1 \neq \cos\varphi_2$. The particular case of the equality $\cos\varphi_1 = \cos\varphi_2$ requires individual study. We shall set $\varphi_1 - \varphi_2 = \varphi$. In so doing, the only one term remains in the left side of Eq. (5)

$$2\sin\varphi \frac{\partial L}{\partial \mathbf{x}} = \sin\varphi \frac{\partial}{\partial x}(G_1 + G_2) + \cos\varphi \frac{\partial}{\partial z}(G_1 - G_2). \quad (17)$$

Integrating Eq. (17) with boundary condition $L(x = 0, z) = L_0(z)$ yields the following solution:

$$\beta(x,z) = \beta(0,z)\left\{\frac{S_1(x,z)S_2(x,z)}{S_1(0,z)S_2(0,z)}\right\}^{1/2} \exp\left\{\frac{\mathrm{cotan}\varphi}{2} \int_0^x \frac{\partial\Omega(x',z)}{\partial z} \mathrm{d}x'\right\} \quad (18)$$

The attenuation coefficient $\alpha(x, z)$ is determined from the equation

$$2\cos\varphi \frac{\partial L}{\partial \mathbf{x}} - 4\alpha = \cos\varphi \frac{\partial L}{\partial \mathbf{x}}\left(G_1 + G_2\right) + \sin\varphi \frac{\partial \Omega}{\partial x}. \quad (19)$$

As a result, determining the partial derivative $\partial L/\partial z = \partial \ln\beta/\partial z$ from Eq. (18) we find

$$\alpha(x, z) = \frac{1}{4}\left\{\cos\varphi \frac{\mathrm{d}R(z)}{\mathrm{d}z} - \sin\varphi \frac{\partial \Omega(x, z)}{\partial x} + \mathrm{D}(x, z)\right\}, \quad (20)$$

$$\mathrm{D}(x, z) = \frac{\cos^2\varphi}{\sin\varphi} \int_0^x \frac{\partial^2 \Omega(x', z)}{\partial z^2} \mathrm{d}x',$$

$$R(z) = 2L_0(z) - \left[G_1(0, z) + G_2(0, z)\right].$$

Equations (18) and (20) determine entirely the solution of the problem of the symmetrical tomographic lidar sounding of the atmosphere ($\varphi_1 = -\varphi_2$). It should be noted that we must know the profile of the backscattering coefficient $b(x, z)$ along the straight line $x = 0$ to reconstruct the solution in symmetrical sounding. It may be the case in which the atmospheric abnormally turbid region under study formed, for instance, due to an industrial emission of aerosol pollutant or gas, is situated to the right of the z axis, so that the optical conditions in the atmosphere outside this region may be described by a certain standard model.

4. Three-beam scheme. In three–beam sounding each point within the region of space under study is sounded from three different directions $\mathbf{n}_i$ (i = 1, 2, 3), characterized by the polar angles φ_i (i = 1, 2, 3). Inverse problem is described by three partial differential equations of the form (4), which may be regarded as the system of linear algebraic equations

$$Ay = g \quad (21)$$

in the unknowns

$$y_1 = \partial L(x, z)/\partial x, \; y_2 = \partial L(x, z)/\partial z, \; y_3 = \alpha(x, z) \quad (22)$$

with the matrix

$$A = \begin{pmatrix} \sin\varphi_1 & \cos\varphi_1 & -2 \\ \sin\varphi_2 & \cos\varphi_2 & -2 \\ \sin\varphi_3 & \cos\varphi_3 & -2 \end{pmatrix} \quad (23)$$

and the vector in the right side

$$g = \left(\frac{\partial G_1}{\partial \mathbf{n}_1}, \frac{\partial G_2}{\partial \mathbf{n}_2}, \frac{\partial G_3}{\partial \mathbf{n}_3}\right)'. \quad (24)$$

By inverting algebraic system of equations (21), one can obtain the attenuation coefficient $\alpha(x, z)$ and logarithmic partial derivatives of the backscattering coefficient $\beta(x, z)$. The obtained solutions have the simplest form for the symmetrical scheme of sounding, when the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are oriented symmetrically about the vector $\mathbf{n}_3$ directed toward the nadir (Fig. 1). In this case the polar angles are taken to be in the following form: $\varphi_1 = -\varphi_2 = \varphi$, $\varphi_3 = 0$. It is easy to show that under these conditions the attenuation coefficient is given by the formula

$$\alpha(x, z) = \frac{1}{4(1 - \cos\varphi)}\left[2\cos\varphi \frac{\partial G_3}{\partial \mathbf{n}_3} - \left(\frac{\partial G_1}{\partial \mathbf{n}_1} + \frac{\partial G_2}{\partial \mathbf{n}_2}\right)\right]. \quad (25)$$

As can be seen from Eq. (25), in three–beam sounding the attenuation coefficient $\alpha(x, z)$ is determined by linear combination of the logarithmic derivatives of lidar signals with respect to the directions of sounding.

Substituting $a(x, z)$ from the obtained equation (25) into the third equation of system (21), which involves only one partial derivative $\text{z}L/\text{z}z$ for $\varphi = 0$ and on integrating, we obtain the following solution for the backscattering coefficient:

$$\beta(x, z) = \left\{\frac{S_3(x, z)^2}{S_1(x, z)S_2(x, z)}\right\}^k \left(S_1(x, z)S_2(x, z)\right)^{1/2} T(x, z), \quad (26)$$

$$T(x, z) = \exp\left\{-k\sin\varphi \int_0^z \frac{\partial \Omega(x, z')}{\partial x} \mathrm{d}z'\right\},$$

where $k = 1/4 \sin^2(\varphi/2)$. It is obvious from comparison of solutions (18) and (20) and (25) for the symmetrical two-beam and three–beam tomographic sounding problems, respectively, that in reconstruction of the spatial distribution of the attenuation coefficient $\alpha(x, z)$ the lidar signal processing can be considerably simplified by introducing the additional sounding channel in the direction $\mathbf{n}_3$. It is clear that this can be obtained at the cost of complication of the experimental scheme. The solution for the backscattering coefficient $\beta(x, z)$ in both cases has analogous structure.

Analysis of the obtained results shows, that the calculation of logarithmic derivatives of a lidar signal with respect to different directions is an important stage of constructing the solutions. Since the logarithmic derivative of a function remains unchanged when the function is multiplied by a constant in processing the lidar signals, this scheme does not require the absolute calibration of lidar signal for reconstruction the attenuation coefficient. In addition, differentiation of functions measured experimentally is an improperly posed problem[7] because of the errors in measurements. Therefore, in developing the applied methods of interpretation of lidar signals based on the obtained formulas, the regularization algorithms are needed to ensure the stability of the calculation of derivatives.

5. Conclusion. The inverse problems of the airborne lidar tomographic sounding of the atmosphere have been formulated and the analytical solutions for the two–beam and three–beam schemes have been obtained. In contrast to the traditional problems of laser sounding, in reconstructing the two–dimensional spatial distributions of the backscattering and attenuation coefficients it is not unnecessary to use the additional *a priori* information regarding the optical characteristics sought for or their spatial structure. Therefore, the methods of airborne tomographic lidar sounding are applicable first of all for optically dense media with high degree of uncertainty and variability of the lidar ratio. The regions with enhanced concentration of aerosol and absorbing gases, formed due to emission of industrial plants, fires and volcanic activity are the examples of such media. Subsequent investigations should be directed to the development of calculational algorithms for solving the inverse problems of lidar tomographic sounding on the basis of the analytical solutions obtained in the present paper taking into account the discrete character of real measurements. Finally, using the developed algorithms it is necessary to evaluate the spatial resolution and accuracy of the inverse problem solutions by the Monte Carlo methods and to elaborate recommendations for the optimal selection of the polar sounding angles for different optical conditions in the atmosphere.

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