METHODS FOR SOLVING THE PHASE PROBLEM IN DIGITAL PROCESSING OF IMAGES. PART II. ANALYTICAL METHODS OF SOLVING THE PHASE PROBLEM

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Conditions that are sufficient for obtaining a unique solution of phase problem are considered. The Hilbert equations were generalized for a two–dimensional case. A possibility of obtaining an analytical solution of the problem is shown for a two–dimensional discrete case. Unambiguity of solution is considered in a two–dimensional continuous case.

As follows from the general analysis of unambiguity of the phase problem solution (Part I) this problem has no unique solution, if an unknown distribution (image) to be reconstructed can be represented (one–dimensional case) or inherently is a convolution of two or more images (two–dimensional case). To provide a unique reconstruction of the image in this case it is necessary to use an additional information. It seems to be most natural to use a preliminary exponential filtration of an image. The exponential function "penetrates" into the convolution integrand and exponentially filtered convolution is equal to the convolution of filtered images. Exponential filtration is also necessary when deriving an analytical solutions with the help of the Hilbert transforms because it shifts the bands of the roots in the complex plane and makes it possible to establish one–to–one relation between the modulus and the phase.

CONDITIONS SUFFICIENT FOR OBTAINING A UNIQUE SOLUTION OF THE PHASE PROBLEM

Let the image $J_1(\mathbf{t})$

$$J_1(\mathbf{t}) = J(\mathbf{t}) \exp(-\gamma\mathbf{t}) , \tag{1}$$

where $\gamma = (\gamma_1, \gamma_2)$ is the filtration vector, be called the exponentially filtered analog (EA) of the image $J(\mathbf{t})$.

Let us consider the simplest case to demonstrate the possibilities of exponential filtration. In a one–dimensional case the Fourier transform $f_1(x)$ of the EA of an image has the form

$$f_1(x) = \hat{F}\{J_1\} = \int\limits_{-\infty}^{+\infty} J(t)\, e^{-\gamma t}\, e^{ixt}\, dt = A_1(x)\, e^{i\varphi_1(x)} , \tag{2}$$

where all designations are the same as in Eq. (1) (Part I). At small $\gamma < 1/S$ we can expand the exponential function into a series and two first terms of which are

$$f_1(x) \approx f(x) + i\gamma \frac{df(x)}{dx} ,$$

where $f(x)$ is the Fourier transform of the image $J(t)$. Equalizing the modules we obtain the approximate relation

$$A_1(x) \approx A(x) \left\{1 - \gamma \frac{d\varphi(x)}{dx}\right\} . \tag{3}$$

So, once two modules of the spectrum $A_1(x)$ and $A(x)$ and the filtration coefficient γ are known one can approximately reconstruct the derivative of the phase $\varphi'(x)$ and, consequently, the phase itself ($\varphi(0) = 0$).

An analogous consideration of a two–dimensional case can be easily carried out. Introducing two EA's of a two–dimensional image as

$$J_2(t_1, t_2) = J(t_1, t_2) \exp(-\gamma_1 t_1) ,$$

$$J_3(t_1, t_2) = J(t_1, t_2) \exp(-\gamma_2 t_2) ,$$

then making their Fourier transforms, and expanding the exponential function into a series ($\gamma_1, \gamma_2 < 1/S$), and finally equalizing the modules we obtain the approximate relations

$$A_2(x_1, x_2) \approx A(x_1, x_2) \left\{1 - \gamma_1 \frac{d\varphi(x_1, x_2)}{dx_1}\right\} ;$$

$$A_3(x_1, x_2) \approx A(x_1, x_2) \left\{1 - \gamma_2 \frac{d\varphi(x_1, x_2)}{dx_2}\right\} . \tag{4}$$

Thus, knowing three modules of the Fourier spectrum A_3, A_2, and A and γ_1, γ_2 we can reconstruct two partial derivatives of the phase φ_{x_1} and φ_{x_2}. Then by integrating and joining these derivatives we can find an approximation of the function $\varphi(x_1, x_2)$. Generalization of these results to the case of arbitrary vectors of filtration we formulate in the form of the following statements.

Statement 1. Let $J(t)$ be the finite function within the region S. Then $J(t)$ can be unambiguously determined by the modulus of its Fourier spectrum and by the modulus of its EA spectrum.[1]

Statement 2. Let $J(\mathbf{t})$ be the finite function within the region S. Then $J(t)$ is completely determined by the modulus of the Fourier spectrum and by the modules of the Fourier spectra of two its EA's, the filtration vectors of which are orthogonal.

The proof of Statement 2 is analogous to the proof of Statement 1. It is based on factorization of the spectrum[1] with respect to each variable.

A more detailed analysis shows that the orthogonality of the filtration vectors is an unnecessary condition.

Statement 3. Let $J(\mathbf{t})$ be the finite function within the region S. Then, $J(\mathbf{t})$ can be unambiguously determined by the Fourier spectrum modulus and the modules of the Fourier

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spectra of two its EA's the filtration vectors of which are noncollinear.[2]

It should be noted that the term exponential filtration can in fact be omitted, since the definition of the Fourier transform of an EA of image (1) completely coincides with the generalized definition of the Laplace transform $L(p)$ (see Ref. 3) for $p = -\gamma + ix$, with the generalized Fourier transform $f(w)$ for $w = x + i\gamma$ (see Part I), and with the definition of the integral exponential function,[4] as well.

However, many different terms can make a confusion, therefore, below we shall follow the above–introduced terminology.

As shown in Part 1, a two–dimensional discrete case of the phase problem can be reduced to a one–dimensional case with the help of a line–by–line (or column–by–column) elongation of the image. Thus, a filtration of the two–dimensional image with the filtration vector, which is transformed into a quasi–continuous one–dimensional filtration during the elongation process, creates conditions corresponding to Statement 1.

Let us consider a technique of a line–by–line elongation without zeroes for a two–dimensional EA of the image of the following form: $J_{n_1,\, n_2}\exp(-\gamma_1 n_1 - \gamma_2 n_2)$.

Let us now put in correspondence to it a one–dimensional EA of the image $I_n\exp(-\gamma_0 n)$ according to the rule

$$J_{n_1,\, n_2}\exp(-\gamma_1 n_1 - \gamma_2 n_2) = I_n\exp(-\gamma_0 n) \text{ at } n = n_1 + n_2(N_1 + 1).$$

By equalizing the arguments of the exponential functions and taking into account the elongation rule we find that $\gamma_1 = \gamma_0$, $\gamma_2 = \gamma_0(N_1 + 1)$.

Thus, the vector of a two–dimensional filtration that is being transformed in a quasi–continuous one–dimensional filtration has the form

$$\gamma = \{\gamma_0,\ (N_1 + 1)\gamma_0\}\ . \qquad (5)$$

Statement 4. A two–dimensional discrete image $J_{n_1,\, n_2}$ is completely determined by the modulus of the Fourier spectrum and modulus of the Fourier spectrum of its EA with the specially chosen filtration vector γ of the form (5) (γ_0 is an arbitrary vector).

THE HILBERT EQUATIONS

An analytical relation of the modulus to the phase in a one–dimensional continuous case can be described by the so–called generalized Hilbert transforms,[5] which can be obtained by applying the Cauchy formula with the integration contour shown in Fig. 1*a*, to the function $\ln f(w)/w$. Since $f(w)$ can contain zeroes in the upper half–plane, $\ln f(w)$ is not an analytical function in this case what makes the Cauchy formula inapplicable to it. However, $f(w)$ can be "corrected" so that its absolute keeps the same everywhere on the real axis ($w = x$), while $f(w) \neq 0$ at $y > 0$, i.e.,

$$f'(w) = f(w)\prod_k \frac{w - w_k^*}{w - w_k}\,,$$

where w_k are the roots of the equation $f(w) = 0$ in the upper half–plane ($y > 0$), while $f'(w)$ is a new function satisfying the conditions $|f'(w)| = |f(w)|$ $\left(\text{because } \left|\dfrac{x - w_k^*}{x - w_k}\right| = 1\right)$ and $f'(w) \neq 0$ at $y > 0$.

As a result we have

$$\varphi(x) = -\frac{1}{\pi}\, v.p. \int\limits_{-\infty}^{+\infty} \frac{\ln A(\xi)}{\xi - x}\mathrm{d}\xi - \sum_k \arg\frac{x - w_k^*}{x - w_k}\,; \qquad (6)$$

$$\ln A(x) = \frac{x}{\pi}\, v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(\xi)\,\mathrm{d}\xi}{\xi(\xi - x)} +$$

$$+ \frac{x}{\pi}\sum_k \arg v.p. \int\limits_{-\infty}^{+\infty} \left(\frac{\xi - w_k^*}{\xi - w_k}\right)\frac{\mathrm{d}\xi}{\xi(\xi - x)}\ . \qquad (7)$$

Equations (6) and (7) were called the generalized Hilbert transforms relating the modulus and phase of the Fourier spectrum of a finite function. Usually, Eq. (6) is called the total phase while the first and the second terms are called the minimum phase and the Blyashke phase, respectively. In this case any of the Blyashke phase components can be changed for the reciprocal one: $\dfrac{x - w_k^*}{x - w_k} \to \dfrac{x - w_k}{x - w_k^*}$ that corresponds to the transfer of a root from the lower half–plane in the upper one (the spectrum modulus does not change in this case). This results in the total number of solutions of the phase problem to be ~ 2^{N-1} where N is the number of the roots of $f(w)$.

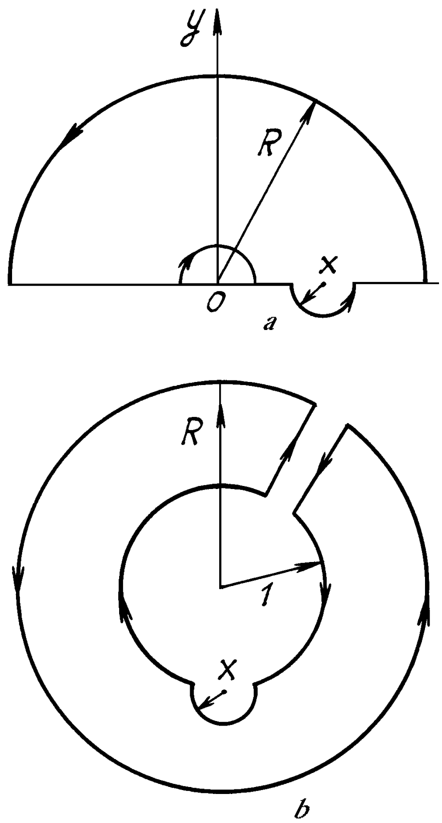

FIG. 1. Integration contour for constructing the Hilbert transforms in the continuous case (a) and in the discrete case (b).

An account of the specific properties of a discrete case means a transition to consideration of z transforms of the image. Application of the Cauchy formula to the function $\ln R_J(z)/z$ under the assumption of the absence of zeroes in the upper half–plane of w–plane (for the z–plane this means the absence of zeroes at $|z| \le 1$), and for the integration contour shown in Fig. 1b, results in[6]

$$\varphi(x) = -\frac{1}{2\pi} v.p. \int\limits_{-\pi}^{\pi} \ln A(\xi) \cot\frac{\xi - x}{2} d\xi \; ; \tag{8}$$

$$\ln A(x) = \frac{1}{2\pi} v.p. \int\limits_{-\pi}^{\pi} \varphi(\xi) \left\{\cot\frac{\xi - x}{2} - \cot\frac{\xi}{2}\right\} d\xi \; . \tag{9}$$

The form of the Blyashke–phase analogs in the discrete case is not of interest, since the ambiguity remains, as previously. The relation between the modulus and the phase becomes unique and analytical, provided that all roots of $f(w)$ are either in one of the half–planes (upper $y > 0$ or lower $y < 0$) or on the real axis ($y = 0$).

Analogously, in the discrete case all roots of $R_J(z)$ have to be either within the circle $|z| \le 1$ or in the region $|z| \ge 1$, or exactly on the circle $|z| = 1$.

Analysis of the behavior of the roots in the discrete case shows that in the w–plane the roots are within a band of a finite width comprising the real axis. The proof of this fact results from the following statement.[7,8]

Statement 5. A finite succession of the form $\Pi(l) = \sum\limits_{n=0}^{N} a_n l^n$ is nonzero for $|l| \le \left(1 + \frac{\max\limits_{n>k+1} |a_n|}{|a_k|}\right)^{-1}$, where a_k is the first nonvanishing element from $\{a_n\}$.

Consequence. For the discrete image J_n the roots $\{w_k\}$ of the equation $f(w) = 0$ are within the band of a finite width parallel to the real x axis. For J_0 and $J_N \ne 0$ the equation of the band is given by the inequalities

$$-\beta_0 \le y \le \alpha_0 \; , \; \forall x; \; \alpha_0 = \ln\left\{1 + \frac{J_{max}}{J_0}\right\} , \; \beta_0 = \ln\left\{1 + \frac{J_{max}}{J_N}\right\} .$$

In the z–plane the roots

$$\left(1 + \frac{J_{max}}{J_0}\right)^{-1} \le |z| \le \left(1 + \frac{J_{max}}{J_N}\right)$$

are in the ring comprising the unit circle.

This consequence makes it possible to find an analytical solution of the phase problem through the minimum–phase relations.

Let us consider the EA of the image $J'_n = J_n \exp(-\alpha_0 n)$. Then,

$$f(w) = \sum_{n=0}^{N} J_n e^{-\alpha_0 n} e^{i(x + iy)n} = \sum_{n=0}^{N} J_n e^{-y'n} e^{ixn} \; ,$$

where $y' = y + \alpha_0$. In this case for $f'(w)$ the real axis x "will be lifted upward" by the value α_0 (the roots remain below). This means that for $y' > 0$ $f'(w) = 0$, i.e., the relation between the modulus and phase can be described either by Eqs. (8) and (9), or by the first terms of Eqs. (6) and (7).

It is natural that for developing such a solution it is necessary to know an *a priori* unknown coefficient α_0 which, however for the typically low–contrast images $\left(\frac{J_{max}}{J_{min}} \sim 100\right)$ can always be taken a little bit greater than its actual value.

Generalization of this result for a two–dimensional case leads to the corresponding generalization of the Hilbert transforms. In a two–dimensional continuous case, first, it is necessary to apply the Cauchy formula, with the integration contour shown in (Fig. 1a), to the function $\frac{\ln f(w_1, 0)}{w_1}$ and then to $\frac{\ln f(x_1, w_2)}{w_2}$ or first to $\frac{\ln f(0, w_2)}{w_2}$ and then to $\frac{\ln f(w_1, x_2)}{w_1}$. For the minimum–phase case we obtain the following relation between the modulus and phase:

$$\varphi(x_1, x_2) = -\frac{x_1}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\ln A(\xi_1, 0)\, d\xi_1}{\xi_1(\xi_1 - x_1)} - \frac{x_2}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\ln A(x_1, x_2)\, d\xi_2}{\xi_2(\xi_2 - x_2)} ; \tag{10}$$

$$\ln A(x_1, x_2) = \frac{x_1}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(\xi_1, 0)\, d\xi_1}{\xi_1(\xi_1 - x_1)} + \frac{x_2}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(x_1, \xi_2)\, d\xi_2}{\xi_2(\xi_2 - x_2)} ;$$

$$\varphi(x_1, x_2) = -\frac{x_2}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\ln A(\xi_2, 0)\, d\xi_2}{\xi_2(\xi_2 - x_2)} - \frac{x_1}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\ln A(x_1, \xi_2)\, d\xi_1}{\xi_1(\xi_1 - x_1)} ;$$

$$\ln A(x_1, x_2) = \frac{x_2}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(0, \xi_2)\, d\xi_2}{\xi_2(\xi_2 - x_2)} + \frac{x_1}{\pi} v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(\xi_1, x_2)\, d\xi_1}{\xi_1(\xi_1 - x_1)} . \tag{11}$$

As can be seen from these equations, the two–dimensional Hilbert transforms are, in fact, a succession of the one–dimensional Hilbert transforms and a difference between Eqs. (10) and (11) is reduced to a difference in the order of successive applications of the one–dimensional Hilbert transforms. Since the finite value of the phase or modulus is independent of the path, the right sides of Eqs. (10) and (11) are equal to each other under the minimum–phase conditions.

The discrete two–dimensional Hilbert transforms are derived for the minimum–phase case analogously by applying the Cauchy formula with the integration contour shown in (Fig. 1b) first to $\ln\frac{R_J(z_1, 1)}{z_1}$ and then to $\ln\frac{R_J(e^{ix_1}, z_2)}{z_2}$ or vice versa.

Discrete analog for Eq. (10) can be given in the following form:

$$\varphi(x_1, x_2) = -\frac{1}{2\pi} v.p. \int\limits_{-\pi}^{\pi} \ln A(\xi, 0) \left\{\cot\frac{\xi_1 - x_1}{2} - \cot\frac{\xi_1}{2}\right\} d\xi_1 -$$

$$-\frac{1}{2\pi} v.p. \int\limits_{-\pi}^{\pi} \ln A(x_1, \xi_2) \left\{\cot\frac{\xi_2 - x_2}{2} - \cot\frac{\xi_2}{2}\right\} d\xi_2 \; ; \tag{12}$$

$$\ln A(x_1, x_2) = \frac{1}{2\pi} v.p. \int_{-\pi}^{\pi} \varphi(\xi_1, 0) \left\{\cot \frac{\xi_1 - x_1}{2} - \cot \frac{\xi_1}{2}\right\} d\xi_1 +$$

$$+ \frac{1}{2\pi} v.p. \int_{-\pi}^{\pi} \varphi(x_1, \xi_2) \left\{\cot \frac{\xi_2 - x_2}{2} - \cot \frac{\xi_2}{2}\right\} d\xi_2 .$$

Let us now try to select the finite values of the filtration coefficients (γ_1, γ_2) of an EA of the image

$$J'_{n_1, n_2} = J_{n_1, n_2} \exp(-\gamma_1 n_1 - \gamma_2 n_2) ,$$

so that the minimum–phase conditions be simultaneously fulfilled in a two–dimensional case $f(w_1, 0) \neq 0$ and $f(x_1, w_2) \neq 0$ $(R_J(z_1, 1) \neq 0, R_J(e^{ix_1}, z_2) \neq 0)$, i.e., let us transfer Statement 5 and its consequence to the two–dimensional case. Detailed theoretical analysis of the conditions of compatibility of the system of inequalities results in the inequality

$$\hat{n}_1 \hat{n}_2 \ln B_1 \ln B_2 \leq 1 , \tag{13}$$

where

$B_1 = \max\limits_{n_2, n_1 > 0} \{J_{n_1, n_2}\} / J_{0, \hat{n}_2}$, $B_2 = \max\limits_{n_2, n_1 > 0} \{J_{n_1, n_2}\} / J_{\hat{n}_1, 0}$, and $J_{0, \hat{n}_2}$ and $J_{\hat{n}_1, 0}$ are either the first nonzero values, or the maximum elements of the image in its first column and the first row, respectively, $\hat{n}_1$ and $\hat{n}_2$ are the coordinates of these elements. Inequality (13) is absolutely true in the following particular cases:

1. $\hat{n}_1 = 0$ (hence, $\hat{n}_2 = 0$) and vice versa.

The image has a corner point $J_{0, 0} \neq 0$. A linear change of variables the cases $J_{0, N_2} \neq 0$, $J_{N_1, 0} \neq 0$ or $J_{N_1, N_2} \neq 0$ are also reduced to this condition. The filtration coefficients are found from the condition

$$\gamma_1 \geq \ln\left\{1 + \max_{n_2, n_1 > 0} (J_{n_1, n_2}) / J_{0, 0}\right\} ,$$

$$\gamma_2 \geq \ln\left\{1 + \max_{n_1, n_2 > 0} (J_{n_1, n_2}) / J_{0, 0}\right\} .$$

2. Either $B_1 < 1$ or $B_2 < 0$ but not simultaneously.

This corresponds to the presence of a bright point on the image edge or to the case is which the energy is concentrated on the image edges.

3. $B_1 \approx B_2 \approx 1$. This condition is fulfilled for images with a very low contrast, or with a constant intensity.

Thus, in contrast to the one–dimensional case in the discrete two–dimensional case we cannot use directly the two–dimensional Hilbert transforms in combination with the exponential filtration, what makes us to reduce first the two–dimensional case to the one–dimensional case. By combining Statements 4 and 5 we obtain for γ_0 from Statement 4 the inequality

$$\gamma_0 \geq \ln\left\{1 + \frac{\max\limits_{n_1, n_2 > 0} (J_{n_1, n_2})}{J_{\hat{n}_1, 0}}\right\} , \tag{14}$$

where $J_{\hat{n}_1, 0}$ is the first nonzero element in the zeroth row of the image.

Statement 6. Let $J(\mathbf{n})\exp(-\boldsymbol{\gamma}\mathbf{n})$ be the EA of a discrete image. Here $\boldsymbol{\gamma}$ is found from Eq. (3) and γ_0 statisfies condition (14). Then, the modulus and phase of the Fourier spectrum of a row–by–row elongated one–dimensional analog of this image are unambiguously related by Eqs. (8) and (9).

If not only the direction but also the value of the filtration vector $\boldsymbol{\gamma}$ is chosen specifically one can construct the exact analytical solution of the phase problem in a two–dimensional case by using the method of reduction of a two–dimensional discrete case to a one–dimensional case.

Let us consider some practical aspects of this solution.

A one–dimensional analog of an EA of a two–dimensional image has the form $I_n \exp(-\gamma_0 n)$, $n = n_1 + n_2(N_1 + 1)$. By substituting the value γ_0 from Eq. (14) in this relation and assuming that, for example, $\frac{J_{max}}{J_{min}} \sim 100$ we obtain $\gamma_0 \sim 10$. If the image arrays are of 64×64 dimensionality, then the final value $n = 4096$ and the last element of the EA of the image has the form $I_{4096} \times 100^{-4096}$, meanwhile the accuracy limits of the modern computers are of 10^{70}–10^{300}. Consequently, this method has a limited application only to small–dimensional images which approximately involve 10×10 resolved elements.

The above–generalized two–dimensional Hilbert transforms make it possible to consider theoretically the question on unambiguity of the phase problem in a two–dimensional continuous case. Below this question is considered in more detail.

QUALITATIVE ANALYSIS OF THE UNAMBIGUITY IN A TWO–DIMENSIONAL CONTINUOUS CASE

If one refuses from the minimum–phase limitations the first relations in Eqs. (10) and (11) take the form

$$\varphi(x_1, x_2) = \varphi_{\min}(x_1, 0) - \arg \sum_k \frac{x_1 - w^*_{1k}}{x_1 - w_{1k}} +$$

$$+ \varphi(\hat{x}_1, x_2) - \arg \sum_l \frac{x_2 - w^*_{2l}(\hat{x}_1)}{x_2 - w_{2l}(\hat{x}_1)} \tag{15}$$

$$\varphi(x_1, x_2) = \varphi_{\min}(0, x_2) - \arg \sum_m \frac{x_2 - w^*_{2m}}{x_2 - w_{2m}} +$$

$$+ \varphi_{\min}(x_1, \hat{x}_2) - \arg \sum_n \frac{x_1 - w^*_{1n}(\hat{x}_2)}{x_1 - w_{1n}(\hat{x}_2)} , \tag{16}$$

where $\{w_{1k}\}$, $\{w_{2l}(\hat{x}_1)\}$, $\{w_{2m}\}$, $\{w_{1m}(\hat{x}_2)\}$ are the roots of equations $f(w_1, 0) = 0$, $f(\hat{x}_1, w_2) = 0$, $f(0, w_2) = 0$, $f(w_1, \hat{x}_2) = 0$, respectively, and designation $\hat{x}_i$, signifies that the corresponding variable is fixed.

As has already been mentioned the final value of the phase must not depend on the path being calculated along and Eq. (15) is equal to Eq. (16) (the so–called closure condition).

Constructions of the solution can appear only at transferring the roots, i.e., when only the second and fourth terms are changed. In this case the constructed solutions have the same region S as actual solutions, since the squared modulus, and hence, the autocorrelation having the size of $2S$ do not change in this case (under conditions the solutions are positive).

Let us assume that a transfer of one of the roots was done along the line x_2 ($x_1 = 0$) and the second term of Eq. (16) was changed (the root w_{2m}^* will be replaced by w_{2m}). In order to keep the equality between Eqs. (15) and (16) it is necessary to change either the fourth term in Eq. (16) or the variable term in Eq. (15). Since the phase at the point x_1, x_2 can be calculated using these equations along any broken line, an infinite set of such broken lines exists in a continuous case while a finite value is also fixed being equal to the sum of the terms at the straight segments of the broken line, we can conclude that the "local" transfer of roots on any segment of a broken line has to be accompanied by the "global" transfer of roots in all other one–dimensional sections to compensate for the local one. As follows from the principle of transfer of the roots, the phase changes $\Delta\varphi$ cannot be infinitely small values since the set of roots is not continuous and is different for each section. Therefore, it is practically impossible to compensate for the local discrete phase change (jump) (i.e., to keep the closure condition) with the help of infinite or finite number of other finite phase jumps.

Thus, the possibility of constructing an extraneous solution in a two–dimensional continuous case can only be realized with the help of a "highly coordinated" "global" transfer of the regions or some closed sets of zeroes in a four–dimensional space of the roots $f(w_1, w_2) = 0$. It is obvious that this can be done only at a very specific disposition of such set, for example, in Ref. 9, where in virtue of nearly circular symmetry of the image a two–dimensional case is reduced to a one–dimensional case in terms of the Bessel functions.

The correctness of the two–dimensional Hilbert equations obtained and the "closure" condition existence are confirmed indirectly by Refs. 10 and 11, in which an approximate method was used for retrieving the phase which makes it possible to retrieve the modules of rare phases in the orthogonal media. In this case, when inverting the phases themselves there appeared an uncertainty in sign, which was eliminated by the train of trials using a constancy of the final phase value irregardless of the ways of coming at this point. As a result, the set of solutions was obtained in a one–dimensional case, since there are no "closure" condition, while in a two–dimensional case it is unique.

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