CALCULATION OF POLARIZATION CHARACTERISTICS OF RADIATION REFLECTED BY A PLANE LAYER OF A TURBID MEDIUM

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The improvement way for vector generalization of small-angle modification of spherical harmonics approximation for backward viewing hemisphere has been expounded. It is based on perturbation theory method. The numerical results of the improvement carried out are adduced. Besides obtained and analyzed is the limit transition from vector small-angle modification of spherical harmonics approximation to small-angle approximation obtained by Dolin and its polarization generalization made by Zege and Chaikovskaya. This fact makes small-angle modification of spherical harmonics approximation to be the general form of small-angle approximation.

The analysis of results of remote sounding of an underlying surface by optical methods can be more effective if one enlists *a priory* information obtained from the calculation of polarization characteristics of radiation reflected by a plane layer of a turbid medium. As was shown in Ref. 1, the small-angle modification of the spherical harmonics method (below, MSH means the small-angle modification of the method of spherical harmonics, SH means spherical harmonics) permits one to calculate the fields of polarization characteristics of radiation with satisfactory accuracy when optical parameters vary widely. But the approximation, first, lays rather rigid restrictions upon the scattering phase matrix of the medium, namely, slow monotonic descending of its angular spectrum when decomposed by generalized spherical functions, and, second, it has considerable error in the backward viewing hemisphere.

Overwhelming majority of natural formations are heterogeneous structures with patches of coarse particles that, according to Mie theory, have scattering phase matrices whose properties are very close to MSH demands.[2,3] In order to improve the solution of the vector equation of radiation transfer (VERT) obtained within the frame of MSH, we represent the scattering phase matrix as a sum of two components

$$\overleftrightarrow{X}(\mu) = a\,\overleftrightarrow{X}_{sa}(\mu) + (1 - a)\,\overleftrightarrow{X}_{p}(\mu)\,, \qquad (1)$$

where $\overleftrightarrow{X}_{sa}(\mu)$ is the small-angle part of the scattering phase matrix with slowly and monotonously descending angular spectrum; $\overleftrightarrow{X}_{p}(\mu)$ is the perturbation of the small-angle matrix having the small number of harmonics in the angular spectrum ($k \le 3 \div 5$); $(1 - a)$ is the small perturbation parameter, $a \lesssim 1$; μ is the cosine of the scattering angle.

For convenience, let us represent VERT in the plane geometry for the Stokes vector-parameter $\mathbf{L}(\tau, \hat{\mathbf{l}})$ in an operator form:

$$\mathbb{D}\,\mathbf{L} = \mathbb{S}\,\mathbf{L}, \qquad (2)$$

where $\mathbb{D} = \mu_0 \dfrac{\partial}{\partial\tau} + \overleftrightarrow{\mathbf{I}}$ is the differential operator of radiation transfer in the small-angle approximation in the vector form[1]; μ_0 is the cosine of the incidence angle of a parallel radiation flux onto a plane layer; $\mu_0 = (\hat{\mathbf{l}}_0\,, \hat{\mathbf{z}})$, $\hat{\mathbf{z}}$ is the unit vector along the OZ axis directed downward normally to the layer boundary; $\tau = \int_0^z \varepsilon(z)\,\mathrm{d}z$ is the optical depth; $\mathbb{S}\mathbf{L} = \int \overleftrightarrow{S}(\mathbf{l}, \mathbf{l}')\,\mathbf{L}(\tau, \hat{\mathbf{l}})\,\mathrm{d}\hat{\mathbf{l}}'$ is the collision integral; $\overleftrightarrow{S}$ is the phase matrix. Other designations are taken from Ref. 1.

Let us complete the equation (2) of the problem of reflection of polarized radiation by a layer of a turbid medium with the boundary conditions

$$\begin{cases} \mathbf{L}(\tau, \hat{\mathbf{l}})\,\Big|_{\Gamma_1} = \mathbf{L}_0\,\delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0), \\ \mathbf{L}(\tau, \hat{\mathbf{l}})\,\Big|_{\Gamma_2} = 0\,, \end{cases} \qquad (3)$$

where $\mathbf{L}_0$ is the vector-parameter of the incident radiation; $\Gamma_1 = \{z = 0,\ \hat{\mathbf{l}} \in \Omega_+\}$; $\Gamma_2 = \{z = H,\ \hat{\mathbf{l}} \in \Omega_-\}$;

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Ω_+ and Ω_- are upper and lower viewing hemispheres respectively ($\Omega_+ \cup \Omega_- = \Omega$); H is the complete width of the layer.

In accordance with Eq. (1), the operator $\mathbb{S}$ in Eq. (2) also can be resolved into two parts:

$$\mathbb{S} = a\,\mathbb{S}_{\rm sa} + (1 - a)\,\mathbb{S}_{\rm p}. \tag{4}$$

We will seek for the solution of the boundary-value problem (2)–(3) as a series of the perturbation theory

$$\mathbf{L}(\tau, \hat{\mathbf{l}}) = \sum_{n=1}^{\infty} (1 - a)^n\, \mathbf{L}_{(n)}(\tau, \hat{\mathbf{l}})\,. \tag{5}$$

Substituting Eqs. (4) and (5) into the VERT (2) we obtain

$$\sum_{n=1}^{\infty} (\mathbb{D} - a\,\mathbb{S}_{\rm sa} - (1 - a)\,\mathbb{S}_{\rm p})\,(1-a)^n\, \mathbf{L}_{(n)}(\tau, \hat{\mathbf{l}}) = 0\,. \tag{6}$$

It follows from Eq. (6) that the coefficients at the same powers of $(1 - a)$ equal zero what is equivalent to the system of recurrent boundary-value problems

$$n = 0: \quad \mathbb{D}\,\mathbf{L}_{(0)} = a\,\mathbb{S}_{\rm sa}\,\mathbf{L}_{(0)}, \tag{7}$$

$$\begin{cases} \mathbf{L}_{(0)}(\tau, \hat{\mathbf{l}})\Big|_{\Gamma_0} = \mathbf{L}_0\,\delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0), \\ \mathbf{L}_{(0)}(\tau, \hat{\mathbf{l}})\Big|_{\Gamma_2} = 0\,; \end{cases}$$

$$n = k \ge 1: \quad \mathbb{D}\,\mathbf{L}_{(k)} = a\,\mathbb{S}_{\rm sa}\,\mathbf{L}_{(k)} + \mathbb{S}_{\rm p}\,\mathbf{L}_{(k-1)}, \tag{8}$$

$$\begin{cases} \mathbf{L}_{(k)}(\tau, \hat{\mathbf{l}})\Big|_{\Gamma_1} = 0\,, \\ \mathbf{L}_{(k)}(\tau, \hat{\mathbf{l}})\Big|_{\Gamma_2} = 0\,; \end{cases}$$

where $\Gamma_0 = \{z = 0,\ \hat{\mathbf{l}} \in \Omega\}$ what is the necessary condition to turn to MSH.[1,4]

It is easy to see that the first boundary-value problem (7) completely satisfies the conditions of MSH applicability and its solution for the case of plane geometry can be written in the form

$$\mathbf{L}_{(0)}(\tau;\nu,\varphi) = \sum_{n=-1}^{+1}\sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\, e^{i2\pi\varphi}\, \overleftrightarrow{Y}{}^{2n}_{l}(\nu)\,\overleftrightarrow{\ell}_l(\tau)\,\mathbf{f}^{2n}_{l}(0)\,, \tag{9}$$

where $\left\{\overleftrightarrow{Y}{}^{2n}_{l}(\nu)\right\}$ are generalized spherical functions[1]; $\nu = (\hat{\mathbf{l}}, \hat{\mathbf{l}}_0)$; $\mathbf{f}^{2n}_{l}(0)$ are the coefficients of expansion of the right-hand side of the upper boundary condition of the boundary-value problem (7) in generalized spherical functions, which have the following form

$$\mathbf{f}^{0}_{l}(0) = \frac{L}{2}\begin{bmatrix} 0 \\ 1-q \\ 1+q \\ 0 \end{bmatrix}, \qquad \mathbf{f}^{2}_{l}(0) = \frac{L}{2}\begin{bmatrix} p \\ 0 \\ 0 \\ p \end{bmatrix} \tag{10}$$

in the reference plane $(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0)$ for the incident radiation with brightness L, degree of polarization p, ellipticity q (Ref. 6) polarization plane coinciding with the plane $(\hat{\mathbf{l}}_0\times\hat{\mathbf{z}})$, $\forall\, l$; $\overleftrightarrow{\ell}_l(\tau)$ is the angular spectrum of the matrix surface Green's function $\overleftrightarrow{\ell}_l(\tau,\, \hat{\mathbf{l}}_0 \to \hat{\mathbf{l}})$ (transfer matrix[6])

$$\mathbf{L}_{(0)}(\tau;\, \hat{\mathbf{l}}) =$$

$$= \int \overleftrightarrow{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0)\,\overleftrightarrow{\ell}(\tau;\hat{\mathbf{l}}'\to\hat{\mathbf{l}})\,\overleftrightarrow{R}(\hat{\mathbf{l}}'\times\hat{\mathbf{l}}_0\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}')\,\mathbf{L}_0\,\delta(\hat{\mathbf{l}}'-\hat{\mathbf{l}}_0)\,d\hat{\mathbf{l}}' \tag{11}$$

where $\overleftrightarrow{R}$ is the rotator, and the transfer matrix being expanded has the form

$$(\overleftrightarrow{\ell}_l(\tau;\, \hat{\mathbf{l}}'\to\hat{\mathbf{l}}))_{rs} = \sum_{l=0}^{\infty} (2l+1)(\overleftrightarrow{\ell}_l(\tau))_{rs}\, P^{l}_{rs}(\hat{\mathbf{l}}',\, \hat{\mathbf{l}})) \tag{12}$$

similar to that of the scattering matrix, as is seen from Eqs. (9) and (11).

The connection between the generalized Legendre polynomials P^{l}_{rs} and $\overleftrightarrow{Y}{}^{2n}_{l}$ is considered in Ref. 1. If Eq. (9) is solved using MSH, one should change the single scattering albedo of the medium Λ by $a\Lambda$ in accordance with Eq. (7).

The boundary-value problem (9) is similar to the problem (7), but there are zero boundary conditions and a source function in the right-hand side of the equation. By definition of the transfer matrix and by virtue of the connection between[7] the surface Green's function and the volume one, the solution to the problem (9) can be written as a superposition integral (for the case $\hat{\mathbf{l}} \in \Omega_-$):

$$\mathbf{L}_{(1)}(\tau;\, \hat{\mathbf{l}}) = \int_{\tau}^{\tau_0}\int \frac{1}{|(\hat{\mathbf{l}}',\, \hat{z})|}\, \overleftrightarrow{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}_0\times\hat{\mathbf{l}})\,\overleftrightarrow{\ell}(\tau'-\tau;\, \hat{\mathbf{l}}'\to\hat{\mathbf{l}}) \times$$

$$\times\, \overleftrightarrow{R}(\hat{\mathbf{l}}'\times\hat{\mathbf{l}}_0\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}')\,\mathbb{S}_{\rm sa}\,\mathbf{L}_0(\tau,\, \hat{\mathbf{l}})\,d\hat{\mathbf{l}}'\,d\tau'. \tag{13}$$

After substituting the expression for the operator $\mathbb{S}_{\rm sa}$ and taking into account the optical reciprocity theorem, we have the following expression for the transfer matrix according to Refs. (7)–(8).

$$\frac{1}{|(\hat{\mathbf{l}}',\hat{z})|}\overset{\leftrightarrow}{\ell}(\tau'-\tau;\ \hat{\mathbf{l}}'\to\hat{\mathbf{l}}) = -\frac{1}{|(\hat{\mathbf{l}},\hat{z})|}\overset{\leftrightarrow}{\ell}(\tau'-\tau;\ -\hat{\mathbf{l}}\to -\hat{\mathbf{l}}'), \tag{14}$$

Then the expression (13) takes the form

$$\mathbf{L}_{(1)}(\tau;\ \hat{\mathbf{l}}) = \frac{\Lambda}{4\pi\eta_0}\times$$

$$\times \int_{\tau}^{\tau_0}\int \overset{\leftrightarrow}{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}_0\times\hat{\mathbf{l}})\ \overset{\leftrightarrow}{\ell}(\tau'-\tau;\ \hat{\mathbf{l}}'\to\hat{\mathbf{l}})\ \overset{\leftrightarrow}{R}(\hat{\mathbf{l}}'\times\hat{\mathbf{l}}_0\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}')\times$$

$$\times\int\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}''\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}_0\times\hat{\mathbf{l}}')\overset{\leftrightarrow}{X}_{\mathrm{sa}}(\hat{\mathbf{l}}'',\hat{\mathbf{l}}')\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}''\times\hat{\mathbf{l}}_0\to\hat{\mathbf{l}}''\times\hat{\mathbf{l}}')$$

$$\mathbf{L}_0(\tau',\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}''\mathrm{d}\hat{\mathbf{l}}'\mathrm{d}\tau'$$

where $\eta_0 = -(\hat{\mathbf{l}},\ \hat{\mathbf{z}})$.

By virtue of the addition theorem for generalized Legendre polynomials,[5] it follows therefrom that

$$\mathbf{L}_{(1)}(\tau;\ \nu,\ \varphi) =$$

$$= \frac{\Lambda}{4\pi\eta_0}\sum_{n=-1}^{+1}\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\,\mathrm{e}^{i2\pi\varphi}\,\overset{\leftrightarrow}{Y}{}^{2n}_{l}(\nu)\overset{\leftrightarrow}{\ell}{}^{(1)}_{l}(\tau)\,\mathbf{f}^{2n}_{l}(0)\ ,$$

where $\overset{\leftrightarrow}{\ell}{}^{(1)}_{l}(\tau) = \int_{\tau}^{\tau_0}\overset{\leftrightarrow}{\ell}_l(\tau'-\tau)\overset{\leftrightarrow}{X}{}^{(l)}_{\mathrm{p}}\overset{\leftrightarrow}{\ell}_l(\tau')\,\mathrm{d}\tau'$ is the angular spectrum of the transfer matrix in the first order of perturbation theory.

According to the solution of VERT by MSH obtained in Ref. 1, the latter integral can be calculated explicitly. As a result, we obtain

$$\overset{\leftrightarrow}{\ell}{}^{(1)}_{l}(\tau) = \frac{\Lambda\mu_0}{4\pi}\sum_{i,j=1}^{+4}\frac{\exp\{-(\tau/\mu_0)(1-\Lambda(\alpha+\Delta_i))\}}{(\eta_0+\mu_0)(1-\alpha\Lambda)-\Lambda(\Delta_i\,\eta_0+\Delta_j\,\mu_0)}\times$$

$$\times\frac{\overset{\leftrightarrow}{V}_i\,\overset{\leftrightarrow}{X}{}^{(l)}_{\mathrm{p}}\,\overset{\leftrightarrow}{V}_j}{4\,\bar{\Delta}_i\,\bar{\Delta}_j}\{1-\exp[-(\tau_0-\tau)/(\eta_0\,\mu_0)((\eta_0+\mu_0)(1-\alpha\Lambda)-$$

$$-\Lambda(\Delta_i\,\eta_0+\Delta_j\,\mu_0))]\}, \tag{15}$$

where $\overset{\leftrightarrow}{V}_i$, α, Δ_i are matrices and coefficients introduced in Ref. 1; $\overset{\leftrightarrow}{X}{}^{(l)}_{\mathrm{p}}$ is the angular spectrum of the matrix $\overset{\leftrightarrow}{X}_{\mathrm{p}}(\mu)$.

The obtained solution (15) similar the zeroth MSH approximation[1] also has the symmetry properties of the exact solution. For brevity, by analogy with Refs. 9–11, we call it quasi-single.

The curves for the degree of linear polarization in the turbid medium thickness in the case of normal incidence ($\mu_0 = 1$) of the natural light onto the layer are depicted in Fig. 1. The viewing angle θ is counted off the OZ axis, $\cos\theta = (\hat{\mathbf{l}},\ \hat{\mathbf{z}})$. The Chandrasekhar solution is used as an exact one. The curve “perturbationB corresponds to the quasi-single approximation. The representation (1) for the Rayleigh scattering phase matrix is realized in accordance with Ref. 1. From the plots shown in the figure it is seen that the quasi-single approximation essentially improves MSH for the back viewing hemisphere and its accuracy is enough to solve many problems of remote sounding of the environment.

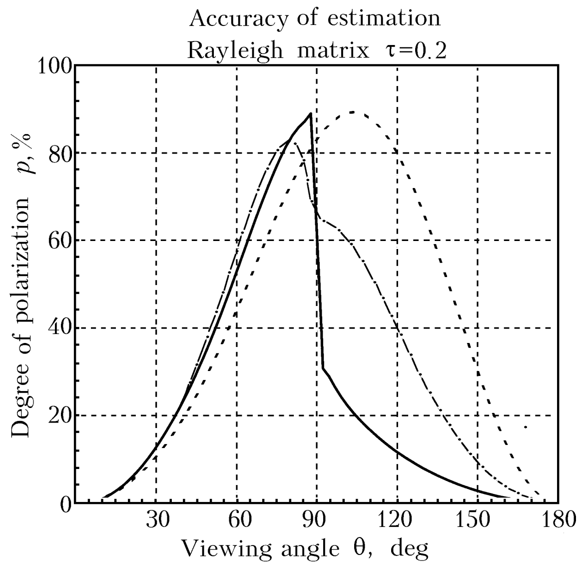

FIG. 1. Comparison of calculation results for degree of polarization of a light field in a plane layer of turbid medium: exact solution (solid curve), “small-angle” solution (dashed curve), “perturbation” (dot-and-dash curve).

The form of the small-angle approximation,[12] including an attempt to generalize it for the vector case,[9-11] can be found in the literature. But this generalization does not describe completely all the properties of partially polarized radiation included in MSH. The connection of the scalar version of MSH with the form[12] is analyzed in Ref. 4. Let us consider the connection between vector generalization of MSH[1] and the approximation from Refs. 9-11. It is based on passing to the limit from the generalized spherical functions to the corresponding Bessel functions $\forall\ H,\ n$ (Ref. 5):

$$\lim_{\nu\to 1} P^{k}_{mn}(\nu) = i^{m-n}\,J_{m-n}(\rho\ l_\perp),\qquad \rho\to\infty\ , \tag{16}$$

where $l_\perp$ is the length of the projection $\mathbf{l}_\perp$ of the unit viewing direction vector $\hat{\mathbf{l}}$ onto the plane orthogonal to the direction of the reference origin of the viewing angle; $\nu = \cos(l_\perp/\rho)$ is the cosine the viewing angle γ; ρ is the length of $\hat{\mathbf{l}}$ equal to the radius of the above-mentioned sphere; $\rho \equiv 1$ but $l_\perp \ll \rho$ when $\nu \to 1$ (i.e. $l_\perp \to \gamma$), and the main condition of passage to the limit is satisfied.

In a similar manner as in Ref. 12 as applied to the vector case for an infinitely wide light beam in a plane-parallel layer, we turn from the vectors $\hat{\mathbf{l}}$ to radius vectors $\mathbf{l}_\perp$ in the CP-representation[13] of polarization and assume that

$$\begin{cases} \mathbf{L}(\tau, \hat{\mathbf{l}}) = \mathbf{L}(\tau, \mathbf{l}_\perp) = \mathbf{L}(\tau, \mathbf{l}_\perp, \varphi); \\ \overleftrightarrow{X}(\hat{\mathbf{l}}', \hat{\mathbf{l}}) \approx \overleftrightarrow{X}(\Delta_\perp), \end{cases} \tag{17}$$

where $\Delta_\perp$ is the length of the vector $\mathbf{l}'_\perp - \mathbf{l}_\perp$ (see Fig. 2). Here, it is convenient to refer the coordinate system to the direction $\hat{\mathbf{l}}_0$ of the radiation incidence onto the layer, i.e., the reference origin of the viewing angle. In this case, the spherical triangle formed by the unit vectors $\hat{\mathbf{l}}'$, $\hat{\mathbf{l}}$, and $\hat{\mathbf{l}}_0$ becomes a usual triangle in the plane $(\mathbf{l}_\perp \times \mathbf{l}'_\perp) \perp \hat{\mathbf{l}}_0$, and the Euler angles[5] χ, $\delta = \arccos\mu = \arccos(\hat{\mathbf{l}}', \hat{\mathbf{l}})$ (scattering angle), χ' become the angles between the sides of the triangle (see Fig. 2), and it is obvious that $\Delta_\perp \to \delta$ for small scattering angles. The rotations on the sphere of infinitely large radius and the motions in the plane become equivalent as well as the corresponding transformation groups.

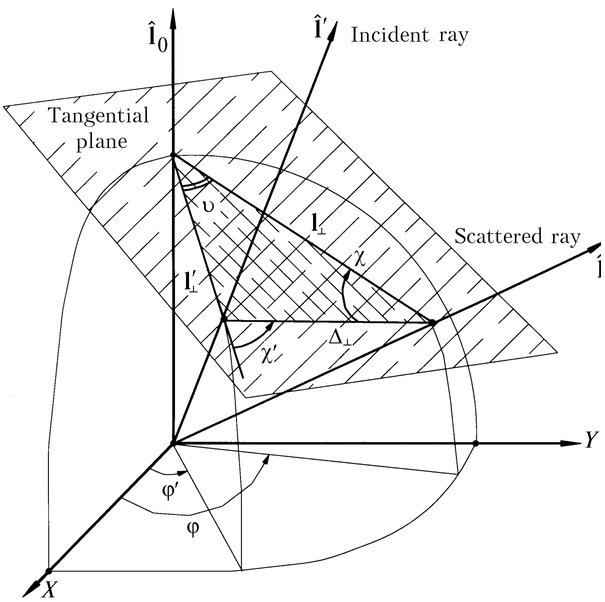

FIG. 2. Change of the variables.

Using the transition to the limit (16) we turn to the matrices

$$\overleftrightarrow{Z}_m(x) = \lim_{\nu \to 1} \overleftrightarrow{Y}^m_k(\nu) =$$

$$= i^m \,\mathrm{Diag}\{-J_{m-2}(x); J_m(x); J_m(x); -J_{m+2}(x)\}, \tag{18}$$

where $. = \rho\, l_\perp$, from the matrices $\overleftrightarrow{Y}^m_l(\nu) = \mathrm{Diag}\{P_{2,m}(\nu); P_{0,m}(\nu); P_{0,m}(\nu); P_{2,m}(\nu)\}$ used to expand the fields $\mathbf{L}$ in the CP-representation.

The assumptions enable us to represent the field of the Stokes vector-parameter (17) as a Hankel transform with the matrix kernel $\overleftrightarrow{Z}_m$ (18):

$$\mathbf{L}(\tau, l_\perp, \varphi) = \sum_{m=-\infty}^{\infty} e^{im\varphi} \frac{1}{2\pi} \int_0^\infty \overleftrightarrow{Z}_m(p\, l_\perp) \mathbf{L}^m(\tau, p)\, p\, dp\,, \tag{19}$$

where $\mathbf{L}^m(\tau, p)$ is the azimuth-spatial spectrum for $\mathbf{L}(\tau, \mathbf{l}_\perp)$.

We also represent the elements of the scattering phase matrix (17) by Hankel transforms of corresponding orders:

$$X_{rs}(\Delta_\perp) = \frac{i^{r-s}}{2\pi} \int_0^\infty J_{r-s}(q\,\Delta_\perp)\, X_{rs}(q)\, q\, dq, \tag{20}$$

where $r, s = 2, 0, 0, -2$ (Ref. 13).

Turning to the matrices $\overleftrightarrow{X}(\Delta_\perp)$ from the matrices $\overleftrightarrow{X}(\hat{\mathbf{l}}', \hat{\mathbf{l}}) = \overleftrightarrow{X}(\mu)$ one should take into account that $\mu \cong 1 - 0.5\,\Delta^2_\gamma$ in the small angle region.

Further, substituting Eqs. (19) and (20) into the integral part of the VERT operator and using simple geometrical relations (see Fig. 2), we have the following expression for the rth element of its column vector

$$\{\mathbb{S}\, \mathbf{L}\}_r = \frac{\Lambda}{4\pi} \int e^{-ir\chi} \sum_s X_{rs}(\Delta_\perp)\, e^{ir\chi'} L_s(\tau, \mathbf{l}'_\perp)\, d^2\mathbf{l}'_\perp =$$

$$= \frac{\Lambda}{4\pi} \frac{1}{(2\pi)^2} \sum_s e^{ir\varphi'} \int_0^{2\pi}\int_0^\infty\int_0^\infty i^{r-s} e^{-ir\chi} J_{r-s}(q\Delta_\perp) e^{ir\chi} X_{rs}(q)\, q\, dq \times$$

$$\times \sum_{m=-\infty}^{\infty} e^{i(m-s)\varphi} \int_0^\infty i^{s-m} J_{s-m}(p l_\perp) L^m_s(\tau, p)\, p\, dp\, l'_\perp\, dl'_\perp\, d\varphi' =$$

$$= \frac{\Lambda}{2(2\pi)^3} \sum_s e^{i(r+s)\varphi} \sum_{m,n} i^{r-m} e^{-in\varphi} \int_0^\infty J_{s-n}(q\, l_\perp) X_{rs}(q)\, dq \times$$

$$\times \int_0^\infty L_s^m(\tau, p) \int_0^\infty J_{r-n}(l'_\perp q)\, J_{s-m}(l'_\perp p)\, l'_\perp\, \mathrm{d}l'_\perp \times$$

$$\times \int_0^{2\pi} \mathrm{e}^{i(m+n-r-s)\varphi'}\, \mathrm{d}\varphi' p \mathrm{d}p = \frac{\Lambda}{4\pi} \sum_{m=-\infty}^{\infty} (-1)^m\ i^{r-m}\ \mathrm{e}^{im\varphi} \times$$

$$\times \frac{1}{2\pi} \int_0^\infty J_{r-m}(q l_\perp) \sum_s X_{rs}(q)\ L_s^m(\tau, q) q \mathrm{d}q$$

or, in the matrix form,

$\mathbb{S}\mathbf{L}(\tau, \mathbf{l}_\perp)=$

$$= \frac{\Lambda}{4\pi} \sum_{m=-\infty}^{\infty} \mathrm{e}^{im\varphi} \frac{(-1)^m}{2\pi} \int_0^\infty \overleftrightarrow{Z}_m(q l_\perp) \overleftrightarrow{X}(q) \mathbf{L}^m(\tau, q) q \mathrm{d}q. \quad (21)$$

The condition of orthogonality for Bessel functions

$$\int_0^\infty J_m(xy)\ J_n(xz) x\ \mathrm{d}x = \frac{(-1)^m}{y}\ \delta_{mn}\ \delta(y - z),$$

where δ_{mn} is the Kronecker delta and $\delta(y - x)$ is the Dirac function, is used in Eq. (21).

For the matrices $\overleftrightarrow{Z}$, this condition can be written in the form

$$\int_0^\infty \overleftrightarrow{Z}_m(xy)\ \overleftrightarrow{Z}_n(xz) \mathrm{d}x = \frac{(-1)^m}{y}\ \delta_{mn}\ \delta(y - z)\ \overleftrightarrow{\mathbf{1}}\ , \quad (22)$$

where $\overleftrightarrow{\mathbf{1}}$= Diag{1; 1; 1; 1}.

The addition theorem for Bessel functions[5] playing a decisive part in deducing Eq. (21), is written in the form

$$\mathrm{e}^{-ir\chi} J_{r-s}(q\Delta_\perp) \mathrm{e}^{is\chi} = \mathrm{e}^{ir\vartheta} \sum_{n=-\infty}^{\infty} J_{r-n}\ (q l'_\perp\ ) J_{s-n}(q l_\perp) \mathrm{e}^{-in\vartheta} \quad (23)$$

in the designations used (see Fig. 2).

Let us transform the differential transfer operator (2) in the “small-angleB form neglecting the summand with the factor $l_\perp^2$ for small scattering angles.[12] Further, substitute Eqs. (19) and (21) into the corresponding parts of the VERT, multiply both the parts by $\overleftrightarrow{Z}_n(p l_\perp)\ \mathrm{e}^{in\varphi}$, and integrate within the admissible values of the variables $l_\perp$, φ. Then, on the basis of the orthogonality condition (22), we obtain the system of four differential equations for the azimuth–spatial spectrum of the matrix Green's function

$$\mu_0 \frac{\partial}{\partial\tau} \overleftrightarrow{\boldsymbol{\ell}}(\tau;\ p) = -\left[\hat{\mathbf{1}} - \frac{1}{4\pi} \overleftrightarrow{X}(p)\right] \overleftrightarrow{\boldsymbol{\ell}}((\tau;\ p)\ . \quad (24)$$

The system (24) is completely identical to the system obtained in MSH, and it is obvious that its solution holds all the properties of the solution by MSH discussed in Ref. 1 and its symmetry.

When neglecting non-diagonal elements in the SP-representation of the matrix $\overleftrightarrow{X}(p)$, one can see that the solution of the system (24) becomes the solution obtained in Refs. 9–11. The full scattering phase matrix (neglecting only the small backscattering) is invariant under small-angle modification of SH and its corollary (24), and, due to CP-representation, all the rotations of the reference plane at every scattering event are accurately taken into account. A purely diagonal matrix describing the attenuation of an incident beam polarization in directions close to its axis is used for zeroth approximation in small-angle methods.[9-11] For instance, it is suitable for solving problems on laser beam transformations. The full scattering matrix enables MSH to describe both the transformation in an event of polarized light scattering and the “generationB of linear polarization by the medium from the natural light due to energy transfer from incoherent part to the coherent one what is essential for solving problems of remote optical sounding.

As to the term Bsmall-angle modification,B it is understood in the sense of small velocity of amplitudes $\overleftrightarrow{\boldsymbol{\ell}}_k(\tau)$ decreasing against the number k. In the scalar version, it is equivalent to the demand of high extension of the brightness body in the region of angle distribution. As to the vector case, extension is characteristic only of the energy component of the Stokes vector-parameter, but the property of the spectrum remains valid for all its components.

Thus, MSH generalized in Ref. 1 for the case of taking into account radiation polarization is the most general form of the small-angle approximation. Application of the quasi-single approximation to MSH permits one to describe the fields of polarization characteristics in all the viewing angles and within sufficiently large limits of variation of medium optical characteristics in an analytically simple form with accuracy sufficient for practice.

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