METHOD OF SPHERICAL HARMONICS: EXACT THREE– DIMENSIONAL MODELS TO COMPUTE DENSITY AND FLUX OF OPTICAL RADIATION IN NATURAL MEDIA

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In order to provide a possibility of studying spectral and energy characteristics of the radiation field of the Earth we formulate exact models for calculating spherical and hemispherical fluxes and densities of optical radiation in natural media that allow for multiple scattering effects. The radiation transfer processes are considered for the case of a 3– D plane layer of a medium with reflecting boundaries assuming both horizontally uniform and nonuniform sources. The models are constructed based only on the method of spherical harmonics. In order to make the models closed we have introduced certain radiation parameters giving their physical interpretation.

Multidimensional models of the optical radiation transfer theory describe radiation processes in natural media (atmosphere, clouds, ocean, hydrometeors) more realistically than the one– dimensional models do. In the majority of problems of remote sounding of the environment use of such models is inevitable.[1] Up-to-date computers allow a versatile software intended for solving different applied problems to be developed. The developed mathematical instrument, in combination with computer technology, provides a basis for methodical and applied researches concerning the problems of evolution and forecast of Earth's energy balance, climate, meteorology, atmospheric photoradiative chemistry, dynamics of ozone layer depletion, transboundary and local transfer of pollution, etc.

The problems with horizontal periodic fluctuations of cloudiness were studied by Romanova.[2] Some results on applying the method of spherical harmonics[3] to solution of such problems for plane media with horizontal inhomogeneities were obtained by Predko, Lebedinskii, Valentyuk, Kozoderov, Mishin, et al. At the Institute of Atmospheric Optics, Tomsk Affiliate of Siberian Branch of the Russian Academy of Sciences, and Computer Center of Siberian Branch of the Russian Academy of Sciences the multidimensional problems of the optical– radiation transfer theory are traditionally solved by Monte Carlo method. The state of the art of foreign researches has been reviewed in Ref. 2. The approximate methods are described in the monograph by Zege, Ivanov, and Katsev.[4]

This paper is aimed at stating a generalized model to calculate the total and hemispherical densities, vertical and horizontal fluxes of optical radiation accounting for multiple scattering in three– dimensional plane media. The radiative characteristics and parameters of radiation are presented via azimuthal and spherical harmonics. The exact mathematical models are constructed using the method of spherical harmonics. Physically meaningful radiation parameters are introduced to make these models closed. Such models are described in Refs. 5– 8 in the approximation of one–dimensional, vertically inhomogeneous scattering and absorbing layer.

In this paper we deal with inhomogeneous three–dimensional layer unbounded horizontally while having finite vertical size with horizontally nonuniform and uniform sources of radiation and reflecting boundaries. The exact models are described with systems of differential equations of the first order and contain nonlinear parameters dependent on moments of radiation intensity. Some exact relations between radiation characteristics are found, which enable one to modify the equations by changing a set of unknown functions.

It should be noted that the problem on closing the system of equations is ambiguous. One of the approaches assumes introduction of nonlinear parameters of radiation. The other approach is an approximate solution of the problems, e.g., in the P_1–approximation of the method of spherical harmonics. Because of the multidimensionality of the problem the radiation parameters can be found by different methods. The areas of applicability of the models proposed will be described in further studies.

STATEMENT OF THE PROBLEM

Consider a scattering and absorbing layer of a medium which is horizontally unbounded ($-\infty < x, y < \infty$) and vertically finite ($0 \le z \le H$). Let it be illuminated with a radiation flux from above or below and let it have reflecting boundary at the top or the bottom. The direction of radiation propagation is determined by the vector $\mathbf{s} = (\mu, \varphi)$, $\mu = \cos\vartheta$, where $\vartheta \in [0, \pi]$ is the zenith angle measured from z axis; the azimuth $\varphi \in [0, 2\pi]$ is measured from x axis. For the downward radiation $s \in \Omega^+ = \{(\mu, \varphi):\ \mu \in [0, 1],\ \varphi \in [0, 2\pi]\}$ and for the upward going radiation $s \in \Omega^- = \{(\mu, \varphi):\ \mu \in [-1, 0],\ \varphi \in [0, 2\pi]\}$; $\Omega = \Omega^+ \cup \Omega^-$. The spatial coordinates are described by a radius–vector $\mathbf{r} = (x, y, z)$ in the layer and $\mathbf{r}_\perp = (x, y)$ in the horizontal plane.

The radiation intensity $\Phi(r, \mu, \varphi)$ is sought as a solution of a boundary–value problem of the transfer theory

$$\begin{cases} (s, \mathrm{grad})\,\Phi(r,\mu,\varphi) + \sigma_t(r)\,\Phi(r,\mu,\varphi) = B(r,\mu,\varphi) + F(r,\mu,\varphi)\,, \\ \Phi\big|_{\Gamma_0} = f_0(r_\perp,\mu,\varphi) + \hat{R}_0\Phi, \quad \Phi\big|_{\Gamma_H} = f_H(r_r,\mu,\varphi) + \hat{R}_H\Phi\,; \end{cases} \tag{1}$$

$$(s,\ \mathrm{grad}) \equiv \mu\,\frac{\partial}{\partial z} + \sin\vartheta\,\cos\varphi\,\frac{\partial}{\partial x} + \sin\vartheta\,\sin\varphi\,\frac{\partial}{\partial y}\,.$$

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The collisional integral

$$B(r,\mu,\varphi) \equiv \frac{\sigma_s(r)}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \Phi(r,\mu',\varphi')\, \gamma(r,\cos\chi)\, d\mu'\, d\varphi' \ ; \quad (2)$$

the scattering phase function is normalized as follows:

$$\frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \gamma(r,\cos\chi)\, d\mu\, d\varphi = \frac{1}{2} \int_{-1}^{1} \gamma(r,\cos\chi)\, d\cos\chi = 1 \ ; \qquad (3)$$

$$\cos\chi = \mu\mu' + \sin\vartheta\, \sin\vartheta' \cos(\varphi - \varphi') \ .$$

In the general case the extinction coefficient $\sigma_t(r) = \sigma_s(r) + \sigma_a(r)$, where $\sigma_s(r)$ is the coefficient of the total aerosol plus molecular scattering, and $\sigma_a(r)$ is the absorption coefficient. The radiation sources $F(r,\mu,\varphi) = F_1(r,s) + F_2(r,s)$, $f_0(r_\perp,\mu,\varphi) = f_{01}(s) + f_{02}(r_\perp, s)$, $f_H(r_\perp,\mu,\varphi) = f_{H1}(s) + f_{H2}(r_\perp, s)$ and the operators $\hat{R}_0\Phi$, $\hat{R}_H\Phi$ describing interaction between the radiation and the boundaries are determined depending on the problem to be solved. For a convenient writing of the boundary conditions we use the sets

$$\Gamma_0 = \{(r,s) \colon z = 0 \ , s \in \Omega^+\} \ , \ \Gamma_H = \{(r,s) \colon z = H \ , s \in \Omega^-\} \ .$$

The even approximation of the continuous solution of the problem (1) given on the sphere at each point as a linear combination of spherical functions[9]

$$\Phi(r,\mu,\varphi) = \sum_{\kappa=0}^{\infty} Y_k(r,\mu,\varphi) \ ,$$
$$Y_\kappa(r,\mu,\varphi) = \sum_{m=0}^{\kappa} \Phi^m_{c\,k}(r)\, C^m_k(\mu,\varphi) + \Phi^m_{s\,k}(r)\, S^m_\kappa(\mu,\varphi) \qquad (4)$$

with the coefficients being the spherical harmonics:

$$\Phi^m_{c\,k}(r) = \frac{2\kappa+1}{2\delta_m \pi} \frac{(\kappa-m)!}{(\kappa+m)!} \int_0^{2\pi} \int_{-1}^{1} \Phi(r,\mu,\varphi)\, C^m_\kappa(\mu,\varphi)\, d\mu\, d\varphi \ ,$$

$\kappa \geq 0$, $0 \leq m \leq \kappa$;

$$\Phi^m_{s\,\kappa}(r) = \frac{2\kappa+1}{2\delta_m \pi} \frac{(\kappa-m)!}{(\kappa+m)!} \int_0^{2\pi} \int_{-1}^{1} \Phi(r,\mu,\varphi)\, S^m_\kappa(\mu,\varphi)\, d\mu\, d\varphi \ ,$$

$\kappa \geq 0$, $0 \leq m \leq \kappa$;

$$\delta_m \equiv 1 + \delta_{m0} \ , \ C^m_\kappa(\mu,\varphi) = P^m_\kappa(\mu) \cos m\varphi \ ,$$

$$S^m_\kappa(\mu,\varphi) = (1 - \delta_{m0})\, P^m_\kappa(\mu) \sin m\varphi \ ;$$

where $P^m_\kappa(\mu)$ are the associated Legendre polynomials; $\delta_{m\kappa}$ is the Kronecker symbol resulting in separation of the variables r, μ, and φ. Using the identity

$$\sum_{\kappa=0}^{\infty} \sum_{m=0}^{\kappa} f^m_{c\,\kappa}\, C^m_\kappa(\mu,\varphi) + f^m_{s\,\kappa}\, S^m_\kappa(\mu,\varphi) =$$
$$= \sum_{m=0}^{\infty} \sum_{\kappa=m}^{\infty} f^m_{s\,\kappa}\, C^m_\kappa(\mu,\varphi) + f^m_{s\,\kappa}\, S^m_\kappa(\mu,\varphi) \ ,$$

in the representation (4) we can separate out the azimuthal dependence

$$\Phi(r,\mu,\varphi) = \sum_{\kappa=0}^{\infty} \sum_{m=0}^{\kappa} \Phi^m_{c\,\kappa}(r) P^m_\kappa(\mu) \cos m\varphi + \Phi^m_{s\,\kappa}(r)\, P^m_k(\mu) \sin m\varphi =$$
$$= \sum_{m=0}^{\infty} \Phi^m_c(r,\mu) \cos m\varphi + \Phi^m_s(r,\mu) \sin m\varphi \ , \qquad (5)$$

where the azimuthal harmonics

$$\Phi^m_c(r,\mu) = \sum_{\kappa=m}^{\infty} \Phi^m_{c\,\kappa}(r)\, P^m_\kappa(\mu) \ ,$$
$$\Phi^m_s(r,\mu) = (1 - \delta_{m0}) \sum_{\kappa=m}^{\infty} \Phi^m_{s\,\kappa}(r)\, P^m_\kappa(\mu) \qquad (6)$$

are determined by the formulas

$$\Phi^m_c(r,\mu) = \frac{1}{\delta_m \pi} \int_0^{2\pi} \Phi(r,\mu,\varphi) \cos m\varphi\, d\varphi \ ,$$

$$\Phi^m_s(r,\mu) = \frac{1}{\delta_m \pi} \int_0^{2\pi} \Phi(r,\mu,\varphi) \sin m\varphi\, d\varphi \ .$$

If the scattering phase function can be represented as an expansion over the Legendre polynomials

$$\gamma(r,\cos\chi) = \sum_{\kappa=0}^{\infty} \omega_\kappa(r)\, P_\kappa(\cos\chi) \ ,$$

$$\omega_\kappa(r) = \frac{2\kappa+1}{2} \int_{-1}^{1} \gamma(r,\cos\chi)\, P_\kappa(\cos\chi)\, d\cos\chi \ ,$$

then using the summation theorem it is possible to separate the angular variables and separate out the azimuthal harmonics[9]

$$\gamma(r,\cos\chi) = \sum_{\kappa=0}^{\infty} \sum_{m=0}^{\kappa} \gamma^m_\kappa(r) \left[C^m_\kappa(\mu,\varphi)\, C^m_\kappa(\mu',\varphi') + \right.$$
$$\left. + S^m_\kappa(\mu,\varphi)\, S^m_\kappa(\mu',\varphi') \right] = \sum_{m=0}^{\infty} \gamma^m(r,\mu,\mu') \cos m(\varphi - \varphi'), \quad (7)$$

$$\gamma^m(r,\mu,\mu') = \sum_{\kappa=m}^{\infty} \gamma^m_\kappa(r)\, P^m_\kappa(\mu)\, P^m_\kappa(\mu') \ ,$$
$$\gamma^m_\kappa(r) = \frac{2}{\delta_m} \frac{(\kappa-m)!}{(\kappa+m)!}\, \omega_\kappa(r) \ . \qquad (8)$$

The azimuthal harmonics can be determined using the integrals[5]:

$$\gamma^0(r,\mu,\mu') = \frac{1}{2\pi} \int_0^{2\pi} \gamma(r,\cos\chi)\, d\varphi = \sum_{\kappa=0}^{\infty} \omega_\kappa(r)\, P_\kappa(\mu)\, P_\kappa(\mu') \ ; \quad (9)$$

$$\gamma^m(r,\mu,\mu') = \frac{1}{\delta_m \pi} \int_0^{2\pi} \gamma(r,\cos\chi) \cos m(\varphi - \varphi')\, d(\varphi - \varphi') \ .$$

Let us assume the following expansions to be true:

$$F(r,\mu,\varphi) = \sum_{\kappa=0}^{\infty} \sum_{m=0}^{\kappa} F^m_{c\,\kappa}(r)\, C^m_\kappa(\mu,\varphi) + F^m_{s\,\kappa}(r)\, S^m_\kappa(\mu,\varphi) =$$
$$= \sum_{m=0}^{\infty} F^m_c(r,\mu) \cos m\varphi + F^m_s(r,\mu) \sin m\varphi \ ; \qquad (10)$$

$$F^m_c(r,\mu) = \sum_{\kappa=m}^{\infty} F^m_{c\,\kappa}(r)\, P^m_\kappa(\mu) \ ,$$
$$F^m_s(r,\mu) = (1 - \delta_{m0}) \sum_{\kappa=m}^{\infty} F^m_{s\,\kappa}(r)\, P^m_\kappa(\mu) \ . \qquad (11)$$

The form of expressions for the expansion coefficients (10) and (11) depends on the specific problems and will be described in a separate issue, where we also discuss the problems related to the boundary conditions.

The method of spherical harmonics is well described in the literature.[3] In our paper we avoid complex representation of spherical functions and negative values of indices what is often used in theoretical investigations but is not convenient in practice. For brevity we omit the arguments (r, μ) in the azimuthal harmonics $F_c^m(r, \mu)$ and $F_s^m(r, \mu)$ and the argument r in the spherical harmonics $F_{c\kappa}^m$, $F_{s\kappa}^m$. We use a mark "↓" for hemispherical characteristics of downward radiation which are determined by integrating over a hemisphere Ω^+ with $\mu > 0$ and a mark "↑" for the hemispherical characteristics of upward going radiation which are determined by integrating over a hemisphere Ω^- with $\mu < 0$.

RADIATION CHARACTERISTICS

The integral (over angles) radiation characteristics, i.e. spherical and hemispherical densities and vertical and horizontal fluxes, are fully determined from zero and first azimuthal harmonics. The spherical characteristics are described by one of the spherical harmonics, whereas hemispherical ones are represented as sums of infinite series expansions over even and odd spherical harmonics. Let us give explicit expressions for radiation characteristics.[10]

The radiation density (an actinometric flux)

$$n(r)=\int\limits_0^{2\pi}\int\limits_{-1}^{1}\Phi(r, \mu, \varphi)\, d\mu\, d\varphi=2\pi\int\limits_{-1}^{1}\Phi_c^0(r, \mu)\, d\mu = 4\pi\,\Phi_{c0}^0(r) ;$$

the density of downward going radiation

$$n^{\downarrow} = \int\limits_0^{2\pi}\int\limits_0^{1}\Phi\, d\mu\, d\varphi = 2\pi\int\limits_0^{1}\Phi_c^0\, d\mu =$$

$$= 2\pi\left[\Phi_{c0}^0 + \frac{1}{2}\Phi_{c1}^0 + \sum_{n=1}^{\infty} t_{2n+1}^0\,\Phi_{c,\,2n+1}^0\right];$$

the density of upward going radiation

$$n^{\uparrow} = \int\limits_0^{2\pi}\int\limits_{-1}^{0}\Phi\, d\mu\, d\varphi = 2\pi\int\limits_{-1}^{0}\Phi_c^0\, d\mu =$$

$$= 2\pi\left[\Phi_{c0}^0 - \frac{1}{2}\Phi_{c1}^0 - \sum_{n=1}^{\infty} t_{2n+1}^0\,\Phi_{c,\,2n+1}^0\right];$$

$$t_{2n+1}^0 \equiv \int\limits_0^{1} P_{2n+1}(\mu)\, d\mu = \frac{(-1)^n\,(2n-1)!!}{2^{n+1}\,(n+1)!}\ ,\ n \ge 0\ ,\ (-1)!! = 1\ ;$$

the vertical radiation flux (along the z axis)

$$J(r) = \int\limits_0^{2\pi}\int\limits_{-1}^{1}\Phi(r, \mu, \varphi)\,\mu\, d\mu\, d\varphi = 2\pi\int\limits_{-1}^{1}\Phi_c^0\,\mu\, d\mu = \frac{4\pi}{3}\Phi_{c1}^0(r) ;$$

the downward vertical radiation flux

$$J^{\downarrow} = \int\limits_0^{2\pi}\int\limits_0^{1}\Phi\,\mu\, d\mu\, d\varphi = 2\pi\int\limits_0^{1}\Phi_c^0\,\mu\, d\mu =$$

$$= \pi\left[\Phi_{c0}^0 + \frac{2}{3}\Phi_{c1}^0 + \frac{1}{4}\Phi_{c2}^0 + 2\sum_{n=2}^{\infty}\beta_{2n}\,\Phi_{c,\,2n}^0\right];$$

and, the upward vertical radiation flux

$$J^{\uparrow} = \int\limits_0^{2\pi}\int\limits_{-1}^{0}\Phi\,\mu\, d\mu\, d\varphi = 2\pi\int\limits_{-1}^{0}\mathrm{F}_c^0\,\mu\, d\mu =$$

$$= -\pi\left[\Phi_{c0}^0 - \frac{2}{3}\Phi_{c1}^0 + \frac{1}{4}\Phi_{c2}^0 + 2\sum_{n=2}^{\infty}\beta_{2n}\,\Phi_{c,\,2n}^0\right];$$

$$\beta_{2n} \equiv \int\limits_0^{1}\mu\, P_{2n}(\mu)\, d\mu = (-1)^{n+1}\,\frac{(2n-3)!!}{2^{n+1}\,(n+1)!}\ ,\ \ n \ge 1\ .$$

It is clear that the total vertical radiation flux

$$J(r) = J^{\downarrow}(r) + J^{\uparrow}(r)\ ;\ \ J^{\downarrow}(r) \ge 0\ ,\ \ J^{\uparrow}(r) \le 0\ .$$

The horizontal radiation flux along the x axis is

$$G_x(r) = \int\limits_0^{2\pi}\int\limits_{-1}^{1}\Phi(r, \mu, \varphi)\,\sin\vartheta\,\cos\varphi\, d\mu\, d\varphi =$$

$$= \pi\int\limits_{-1}^{1}\Phi_c^1\,\sin\vartheta\, d\mu = \frac{4\pi}{3}\Phi_{c1}^1\ ;$$

the downward horizontal radiation flux along the x axis is

$$G_x^{\downarrow} = \int\limits_0^{2\pi}\int\limits_0^{1}\Phi\,\sin\vartheta\,\cos\varphi\, d\mu\, d\varphi = \pi\int\limits_0^{1}\Phi_c^1\,\sin\vartheta\, d\mu =$$

$$= \pi\left[\frac{2}{3}\Phi_{c1}^1 + \frac{3}{4}\Phi_{c2}^1 + \sum_{n=2}^{\infty}\beta_{2n}^1\,\Phi_{c,\,2n}^1\right];$$

and, the upward horizontal radiation flux along the x axis is

$$G_x^{\uparrow} = \int\limits_0^{2\pi}\int\limits_{-1}^{0}\Phi\,\sin\vartheta\,\cos\varphi\, d\mu\, d\varphi = \pi\int\limits_{-1}^{0}\mathrm{F}_c^1\,\sin\vartheta\, d\mu =$$

$$= \pi\left[\frac{2}{3}\Phi_{c1}^1 - \frac{3}{4}\Phi_{c2}^1 - \sum_{n=2}^{\infty}\beta_{2n}^1\,\Phi_{c,\,2n}^1\right].$$

The horizontal radiation flux along the y axis is

$$G_y(r) = \int\limits_0^{2\pi}\int\limits_{-1}^{1}\Phi(r, \mu, \varphi)\,\sin\vartheta\,\sin\varphi\, d\mu\, d\varphi =$$

$$= \pi\int\limits_{-1}^{1}\Phi_s^1\,\sin\vartheta\, d\mu = \frac{4\pi}{3}\Phi_{s1}^1\ ;$$

the downward horizontal radiation flux along the y axis is

$$G_y^{\downarrow} = \int\limits_0^{2\pi}\int\limits_0^{1}\Phi\,\sin\vartheta\,\cos\varphi\, d\mu\, d\varphi = \pi\int\limits_0^{1}\Phi_s^1\,\sin\vartheta\, d\mu =$$

$$= \pi\left[\frac{2}{3}\Phi_{s1}^1 + \frac{3}{4}\Phi_{s2}^1 + \sum_{n=2}^{\infty}\beta_{2n}^1\,\Phi_{s,\,2n}^1\right];$$

and, the upward horizontal radiation flux along the y axis is

$$G_y^{\uparrow} = \int\limits_0^{2\pi}\int\limits_{-1}^{0}\Phi\,\sin\vartheta\,\cos\varphi\, d\mu\, d\varphi = \pi\int\limits_{-1}^{0}\Phi_s^1\,\sin\vartheta\, d\mu =$$

$$= \pi\left[\frac{2}{3}\Phi_{s1}^1 - \frac{3}{4}\Phi_{s2}^1 - \sum_{n=2}^{\infty}\beta_{2n}^1\,\Phi_{s,\,2n}^1\right];$$

$$\beta_{2n}^1 \equiv \int\limits_0^{1} P_{2n}^1(\mu)\,\sin\vartheta\, d\mu = \frac{(-1)^{n+1}\,(2n+1)!!}{2^n\,(2n-1)\,(n+1)\,(n-1)!}\ ,\ n \ge 1\ .$$

The horizontal radiation flux in the azimuthal plane $\varphi = \varphi_{\perp}$ is

$$G_{\perp}(r) = \int\limits_0^{2\pi}\int\limits_{-1}^{1}\Phi(r, \mu, \varphi)\,\sin\vartheta\,\cos(\varphi - \varphi_{\perp})\, d\mu\, d\varphi =$$

$$= \pi\cos\varphi_{\perp}\int\limits_{-1}^{1}\Phi_c^1\,\sin\vartheta\, d\mu + \pi\sin\varphi_{\perp}\int\limits_{-1}^{1}\Phi_s^1\,\sin\vartheta\, d\mu =$$

$$= G_x\cos\varphi_{\perp} + G_y\sin\varphi_{\perp}\ ;$$

the downward horizontal radiation flux in the azimuthal plane $\varphi = \varphi_{\perp}$ is

$$G_{\perp}^{\downarrow}(r) = \int\limits_0^{2\pi}\int\limits_0^{1}\Phi(r, \mu, \varphi)\,\sin\vartheta\,\cos(\varphi - \varphi_{\perp})\, d\mu\, d\varphi =$$

$= G_x^{\downarrow}\cos\varphi_{\perp} + G_y^{\downarrow}\sin\varphi_{\perp}$;

and, the upward horizontal radiation flux in the azimuthal plane $\varphi = \varphi_{\perp}$ is

$$G_{\perp}^{\uparrow}(r) = \int_0^{2\pi}\int_{-1}^{0} \Phi(r, \mu, \varphi)\sin\vartheta\cos(\varphi - \varphi_{\perp})\,d\mu\,d\varphi =$$

$$= G_x^{\uparrow}\cos\varphi_{\perp} + G_y^{\uparrow}\sin\varphi_{\perp}\ .$$

It is clear that $G_x = G_x^{\downarrow} + G_x^{\uparrow}$, $G_y = G_y^{\downarrow} + G_y^{\uparrow}$, $G_{\perp} = G_{\perp}^{\downarrow} + G_{\perp}^{\uparrow}$.

RADIATION PARAMETERS

The coefficients in equations describing the mathematical model for calculating the densities and fluxes, i.e. radiation parameters, are introduced using azimuthal and spherical harmonics.

The K—integral, or the second order moment, is

$$K(r) \equiv \int_0^{2\pi}\int_{-1}^{1} \Phi(r, \mu, \varphi)\,\mu^2\,d\mu\,d\varphi = 2\pi\int_{-1}^{1}\Phi_c^0\,\mu^2\,d\mu =$$

$$= \frac{4\pi}{3}\Phi_{c0}^0 + \frac{8\pi}{15}\Phi_{c2}^0\ ;$$

the hemispherical K—integrals are

$$K^{\downarrow} \equiv \int_0^{2\pi}\int_0^1 \Phi\,\mu^2\,d\mu\,d\varphi = 2\pi\int_0^1 \Phi_c^0\,\mu^2\,d\mu =$$

$$= 2\pi\left[\frac{1}{3}\Phi_{c0}^0 + \frac{1}{4}\Phi_{c1}^0 + \frac{2}{15}\Phi_{c2}^0 + \sum_{n=1}^{\infty} t_{2n+1}\,\Phi_{c,\,2n+1}^0\right];$$

$$K^{\uparrow}(r) \equiv \int_0^{2\pi}\int_{-1}^0 \Phi\,\mu^2\,d\mu\,d\varphi = 2\pi\int_{-1}^0 \Phi_c^0\,\mu^2\,d\mu =$$

$$= 2\pi\left[\frac{1}{3}\Phi_{c0}^0 - \frac{1}{4}\Phi_{c1}^0 + \frac{2}{15}\Phi_{c2}^0 - \sum_{n=1}^{\infty} t_{2n+1}\,\Phi_{c,\,2n+1}^0\right];$$

$$t_{2n+1} \equiv \int_0^1 \mu^2 P_{2n+1}(\mu)\,d\mu = \frac{(-1)^{n+1}\,(2n-3)!!}{2^{n+1}\,(n+1)!}\ ,\quad n \ge 1\ .$$

The coefficient of radiation diffusion along vertical direction is

$$D(r) \equiv K(r)/n(r) = \frac{1}{3} + \frac{2}{15}\Phi_{c2}^0\ /\ \Phi_{c0}^0\ ;$$

the hemispherical vertical coefficients of diffusion are

$$D^{\downarrow}(r) = K^{\downarrow}(r)/n^{\downarrow}(r)\ ;\quad D^{\uparrow}(r) = K^{\uparrow}(r)\ /\ n^{\uparrow}(r)\ .$$

The coefficient of radiation diffusion in the horizontal plane is

$$D_{\perp}(r) \equiv 1 - D(r) = \frac{2}{3}\left[1 - \frac{1}{5}\Phi_{c2}^0\ /\ \Phi_{c0}^0\right];$$

for downward and upward going radiation the coefficients of diffusion in the horizontal plane are

$$D_{\perp}^{\downarrow}(r) \equiv 1 - D^{\downarrow}(r)\ ,\quad D_{\perp}^{\uparrow}(r) \equiv 1 - D^{\uparrow}(r)\ .$$

For a characteristic of angular anisotropy of radiation we introduce the parameters: "mean cosines" and "mean sines".

The mean cosines of the vertical radiation fluxes are

$$\overline{\mu} \equiv J(r)/n(r) = \frac{1}{3}\Phi_{c1}^0/\Phi_{c0}^0\ ;\ \mu^{\downarrow} \equiv J^{\downarrow}/n^{\downarrow} \ge 0\ ;$$
$$\mu^{\uparrow} \equiv J^{\uparrow}/n^{\uparrow} \le 0\ .$$

Let us now introduce the moments of radiation in the horizontal plane

$$\mu_{\perp c}(r) \equiv \int_0^{2\pi}\int_{-1}^1 \mu\,\Phi\sin\vartheta\cos\varphi\,d\mu\,d\varphi = \pi\int_{-1}^1 \mu\sin\vartheta\,\Phi_c^1\,d\mu = \frac{4\pi}{5}\Phi_{c2}^1;$$

$$\mu_{\perp s}(r) \equiv \int_0^{2\pi}\int_{-1}^1 \mu\,\Phi\sin\vartheta\cos\varphi\,d\mu\,d\varphi = \pi\int_{-1}^1 \mu\sin\vartheta\,\Phi_s^1\,d\mu = \frac{4\pi}{5}\Phi_{s2}^1;$$

for downward radiation

$$\mu_{\perp c}^{\perp} \equiv \int_0^{2\pi}\int_0^1 \mu\,\Phi\sin\vartheta\cos\varphi\,d\mu\,d\varphi = \pi\int_0^1 \mu\sin\vartheta\,\Phi_c^1\,d\mu = \pi\left\{\frac{1}{4}\Phi_{c1}^1 + \frac{2}{5}\Phi_{c2}^1 + \sum_{\kappa=3}^{\infty}\Phi_{c\,\kappa}^1\,\beta_{2n}^1\left[\frac{\kappa}{2\kappa+1}\delta_{\kappa+1,\,2n} + \frac{\kappa+1}{2\kappa+1}\delta_{\kappa-1,\,2n}\right]\right\};$$

$$\mu_{\perp s}^{\downarrow} \equiv \int_0^{2\pi}\int_0^1 \mu\,\Phi\sin\vartheta\sin\varphi\,d\mu\,d\varphi = \pi\int_0^1 \mu\sin\vartheta\,\Phi_s^1\,d\mu = \pi\left\{\frac{1}{4}\Phi_{s1}^1 + \frac{2}{5}\Phi_{s2}^1 + \sum_{\kappa=3}^{\infty}\mathrm{F}_{s\,\kappa}^1\,\beta_{2n}^1\left[\frac{\kappa}{2k+1}\delta_{\kappa+1,\,2n} + \frac{\kappa+1}{2\kappa+1}\delta_{\kappa-1,\,2n}\right]\right\};$$

for upward radiation

$$\mu_{\perp c}^{\uparrow} \equiv \int_0^{2\pi}\int_{-1}^0 \mu\,\Phi\sin\vartheta\cos\varphi\,d\mu\,d\varphi = \pi\int_{-1}^0 \mu\sin\vartheta\,\Phi_c^1\,d\mu = -\pi\left\{\frac{1}{4}\Phi_{c1}^1 - \frac{2}{5}\Phi_{c2}^1 + \sum_{\kappa=3}^{\infty}\Phi_{c\,\kappa}^1\,\beta_{2n}^1\left[\frac{\kappa}{2\kappa+1}\delta_{\kappa+1,\,2n} + \frac{\kappa+1}{2\kappa+1}\delta_{\kappa-1,\,2n}\right]\right\};$$

$$\mu_{\Gamma s}^{\uparrow} \equiv \int_0^{2\pi}\int_{-1}^0 \mu\,\Phi\sin\vartheta\sin\varphi\,d\mu\,d\varphi = \pi\int_{-1}^0 \mu\sin\vartheta\,\Phi_s^1\,d\mu = -\pi\left\{\frac{1}{4}\Phi_{s1}^1 - \frac{2}{5}\Phi_{s2}^1 + \sum_{\kappa=3}^{\infty}\Phi_{s\,\kappa}^1\,\beta_{2n}^1\left[\frac{\kappa}{2\kappa+1}\delta_{\kappa+1,\,2n} + \frac{\kappa+1}{2\kappa+1}\delta_{\kappa-1,\,2n}\right]\right\}.$$

The angular anizontropy of horizontal fluxes of radiation along the x and y axes is described by mean cosines:

$$\mu_x(r) \equiv \mu_{\perp c}(r)/G_x(r) = \frac{3}{5}\Phi_{c2}^1\ /\ \Phi_{c1}^1\ ;$$

$$\mu_y(r) \equiv \mu_{\perp s}(r)/G_y(r) = \frac{3}{5}\Phi_{s2}^1\ /\ \Phi_{s1}^1\ ;$$

$$\mu_x^{\downarrow} \equiv \mu_{\perp c}^{\downarrow}/G_x^{\downarrow}\ ;\quad \mu_y^{\downarrow} \equiv \mu_{\perp s}^{\downarrow}/G_y^{\downarrow}\ ;\ \mu_x^{\uparrow} \equiv \mu_{\perp c}^{\uparrow}/G_x^{\uparrow}\ ;\quad \mu_y^{\uparrow} \equiv \mu_{\perp s}^{\uparrow}/G_y^{\uparrow}$$

and mean sines

$$s_x(r) \equiv G_x(r)\ /\ n(r) = \frac{1}{3}\Phi_{c1}^1\ /\ \Phi_{c0}^0\ ;$$

$$s_y(r) \equiv G_y(r)\ /\ n(r) = \frac{1}{3}\Phi_{s1}^1\ /\ \Phi_{c0}^0\ ;$$

$$s_x^{\downarrow} \equiv G_x^{\downarrow}\ /\ n^{\downarrow}\ ;\ s_y^{\downarrow} \equiv G_y^{\downarrow}\ /\ n^{\downarrow}\ ;\ s_x^{\uparrow} \equiv G_x^{\uparrow}\ /\ n^{\uparrow}\ ;\ s_y^{\uparrow} \equiv G_y^{\uparrow}\ /\ n^{\uparrow}\ .$$

Let us introduce the parameters determining the relations between the vertical and horizontal fluxes:

$$c_x(r) \equiv G_x(r)/J(r) = \Phi^1_{c1} \,/\, \Phi^0_{c1} \;;$$
$$c_y(r) \equiv G_y(r)/J(r) = \Phi^1_{s1} \,/\, \Phi^0_{c1} \;;$$
$$c^{\downarrow}_x \equiv G^{\downarrow}_x/J^{\downarrow} \;;\; c^{\downarrow}_y \equiv G^{\downarrow}_y/J^{\downarrow} \;;\; c^{\uparrow}_x \equiv G^{\uparrow}_x/J^{\uparrow} \;;\; c^{\uparrow}_y \equiv G^{\uparrow}_y/J^{\uparrow} \;.$$

AZIMUTHAL HARMONICS OF RADIATION PARAMETERS

To make the mathematical models closed, we use the second azimuthal harmonics of the coefficients of diffusion in the horizontal plane, i.e. the radiation parameters

$$D_{\perp c2}(r) \equiv K_{\perp c2} \,/\, n = \frac{4}{5}\,\mathrm{F}^2_{c2} \,/\, \mathrm{F}^0_{c0} \;;$$
$$D_{\perp s2}(r) \equiv K_{\perp s2} \,/\, n = \frac{4}{5}\,\mathrm{F}^2_{s2} \,/\, \mathrm{F}^0_{c0} \;;$$
$$D^{\downarrow}_{\perp c2}(r) \equiv K^{\downarrow}_{\perp c2} \,/\, n^{\downarrow} \;;\; D^{\uparrow}_{\perp c2}(r) \equiv K^{\uparrow}_{\perp c2} \,/\, n^{\uparrow} \;;$$
$$D^{\downarrow}_{\perp s2}(r) \equiv K^{\downarrow}_{\perp s2} \,/\, n^{\downarrow} \;;\; D^{\uparrow}_{\perp s2}(r) \equiv K^{\uparrow}_{\perp s2} \,/\, n^{\uparrow} \;.$$

The second azimuthal harmonics of K–integral in the horizontal plane are represented vs. the azimuthal and spherical harmonics of intensity

$$K_{\perp c2} \equiv \int\limits_0^{2\pi}\int\limits_{-1}^{1} \sin^2\vartheta\; \Phi(r,\mu,\varphi)\cos 2\varphi\, d\mu\, d\varphi =$$
$$= \pi\int\limits_{-1}^{1} \sin^2\vartheta\; \Phi^2_c(r,\mu)\, d\mu = \frac{16}{5}\,\pi\,\Phi^2_{c2} \;;$$

$$K_{\perp s2} \equiv \int\limits_0^{2\pi}\int\limits_{-1}^{1} \sin^2\vartheta\; \Phi(r,\mu,\varphi)\sin 2\varphi\, d\mu\, d\varphi =$$
$$= \pi\int\limits_{-1}^{1} \sin^2\vartheta\; \Phi^2_s(r,\mu)\, d\mu = \frac{16}{5}\,\pi\,\Phi^2_{s2} \;;$$

$$K^{\downarrow}_{\perp c2} \equiv \int\limits_0^{2\pi}\int\limits_{0}^{1} \sin^2\vartheta\; \Phi\cos 2\varphi\, d\mu\, d\varphi = \pi\int\limits_0^1 \sin^2\vartheta\;\Phi^2_c\, d\mu =$$
$$= \frac{8}{5}\,\pi\,\Phi^2_{c2} + \pi\sum_{n=1}^{\$} b_n\,\Phi^2_{c,\,2n+1} \;;$$

$$K^{\uparrow}_{\perp c2} \equiv \int\limits_0^{2\pi}\int\limits_{-1}^{0} \sin^2\vartheta\; \Phi\cos 2\varphi\, d\mu\, d\varphi = \pi\int\limits_{-1}^0 \sin^2\vartheta\;\mathrm{F}^2_c\, d\mu =$$
$$= \frac{8}{5}\,\pi\,\Phi^2_{c2} - \pi\sum_{n=1}^{\exists} b_n\,\Phi^2_{c,\,2n+1} \;;$$

$$b_n \equiv \frac{1}{3}\int\limits_0^1 P^2_{2n+1}(\mu)\,P^2_2(\mu)\,d\mu = (-1)^{n+1}\frac{(2n+3)!!}{2^n\,(2n-1)(n+2)(n-1)!} \,,$$
$n \geq 2$.

The expressions for integrals $K^{\downarrow}_{\perp s2}$ and $K^{\uparrow}_{\perp s2}$ coincide with those for $K^{\downarrow}_{\perp c2}$ and $K^{\uparrow}_{\perp c2}$ if the subscript "c" is changed for "s" and cos2φ is changed for sin2φ.

BACKSCATTERING CHARACTERISTICS

To close the mathematical models for calculating downward and upward densities as well as vertical and horizontal fluxes of radiation we introduce the backscattering characteristics which depend on properties of the scattering phase function in the back hemisphere.

From the condition of normalization of the scattering phase function (3) we have

$$\gamma_0(r,\mu) \equiv \int\limits_{-1}^{1} \gamma^0(r,\mu,\mu')\, d\mu' = \gamma^+_0(r,\mu) + \gamma^-_0(r,\mu) = 2 \;;$$

$$\gamma^+_0(r,\mu) \equiv \int\limits_{0}^{1} \gamma^0(r,\mu,\mu')\, d\mu' =$$
$$= 1 + \frac{1}{2}\,\omega_1\,\mu + \sum_{m=1}^{\infty} \omega_{2m+1}\,P_{2m+1}\,t^0_{2m+1} \;;$$

$$\gamma^-_0(r,\mu) \equiv \int\limits_{-1}^{0} \gamma^0(r,\mu,\mu')\, d\mu' =$$
$$= 1 - \frac{1}{2}\,\omega_1\,\mu - \sum_{m=1}^{\infty} \omega_{2m+1}\,P_{2m+1}\,t^0_{2m+1} \;.$$

As in the case with a one–dimensional plane layer we introduce the backscattering characteristics

$$\gamma^{\downarrow}_0(r) \equiv \Gamma^{\downarrow}_0(r) \,/\, n^{\downarrow}(r) = 1 - \frac{\omega_1(r)}{2}\,\mu^{\downarrow}(r) - M^{\downarrow}(r) \;;$$
$$\gamma^{\uparrow}_0(r) \equiv \Gamma^{\uparrow}_0(r) \,/\, n^{\uparrow}(r) = 1 + \frac{\omega_1(r)}{2}\,\mu^{\uparrow}(r) - M^{\uparrow}(r) \;,$$

where

$$\Gamma^{\downarrow}_0(r) \equiv 2\pi\int\limits_0^1 \Phi^0_c(r,\mu)\, d\mu \int\limits_{-1}^0 \gamma^0(r,\mu,\mu')\, d\mu' =$$
$$= 2\pi\int\limits_0^1 \Phi^0_c(r,\mu)\,\gamma^-_0(r,\mu)\, d\mu \;;$$

$$\Gamma^{\uparrow}_0(r) \equiv 2\pi\int\limits_{-1}^0 \Phi^0_c(r,\mu)\, d\mu \int\limits_{0}^1 \gamma^0(r,\mu,\mu')\, d\mu' =$$
$$= 2\pi\int\limits_{-1}^0 \Phi^0_c(r,\mu)\,\gamma^+_0(r,\mu)\, d\mu \;;$$

$$M^{\downarrow}(r) \equiv \sum_{m=1}^{\infty} \omega_{2m+1}\,t^0_{2m+1}\int\limits_0^1 \Phi^0_c\,P_{2m+1}\, d\mu \,/\, \int\limits_0^1 \Phi^0_c\, d\mu \,,$$
$$M^{\uparrow}(r) \equiv \sum_{m=1}^{\infty} \omega_{2m+1}\,t^0_{2m+1}\int\limits_{-1}^0 \Phi^0_c\,P_{2m+1}\, d\mu \,/\, \int\limits_{-1}^0 \Phi^0_c\, d\mu \,.$$

Using the moments of zero azimuthal harmonics of the scattering phase function related by the expression

$$\gamma_1(r,\mu) \equiv \int\limits_{-1}^{1} \mu'\,\gamma^0(r,\mu,\mu')\, d\mu' = \gamma^+_1(r,\mu) + \gamma^-_1(r,\mu) = \frac{2}{3}\,\omega_1(r)\,\mu \;;$$

$$\gamma^+_1(r,\mu) \equiv \int\limits_{0}^{1} \mu'\,\gamma^0(r,\mu,\mu')\, d\mu' =$$
$$= \frac{1}{2} + \frac{\omega_1(r)}{3}\,\mu + \sum_{m=1}^{\infty} \omega_{2m}(r)\,P_{2m}(\mu)\,\beta_{2m} \,,$$

$$\gamma^-_1(r,\mu) \equiv \int\limits_{-1}^{0} \mu'\,\gamma^0(r,\mu,\mu')\, d\mu' =$$

$$= -\frac{1}{2} + \frac{\omega_1(r)}{3}\,\mu - \sum_{m=1}^{\infty} \omega_{2m}(r)\, P_{2m}(\mu)\, \beta_{2m}\,,$$

we introduce the backscattering characteristics

$$\gamma_1^{\downarrow}(r) \equiv \Gamma_1^{\downarrow}(r) \,/\, n^{\downarrow}(r) = -\frac{1}{2} + \frac{\omega_1}{3}\,\mu^{\downarrow} - \frac{\omega_2}{16}\,(3D^{\downarrow} - 1) - N_2^{\downarrow}\,;$$

$$\gamma_1^{\uparrow}(r) \equiv \Gamma_1^{\uparrow}(r) \,/\, n^{\uparrow}(r) = \frac{1}{2} + \frac{\omega_1}{3}\,\mu^{\uparrow} + \frac{\omega_2}{16}\,(3D^{\uparrow} - 1) + N_2^{\uparrow}\,,$$

where the moments of radiation intensity, depending on the scattering phase function, are

$$\Gamma_1^{\downarrow}(r) \equiv 2\pi \int_0^1 \Phi_c^0(r, \mu)\, d\mu \int_{-1}^0 \mu'\, \gamma^0(r, \mu, \mu')\, d\mu' =$$

$$= 2\pi \int_0^1 \gamma_1^-(r, \mu)\, \Phi_c^0(r, \mu)\, d\mu\,;$$

$$\Gamma_1^{\uparrow}(r) \equiv 2\pi \int_{-1}^0 \Phi_c^0(r, \mu)\, d\mu \int_0^1 \mu'\, \gamma^0(r, \mu, \mu')\, d\mu' =$$

$$= 2\pi \int_{-1}^0 \gamma_1^+(r, \mu)\, \Phi_c^0(r, \mu)\, d\mu\,;$$

$$N_2^{\downarrow}(r) \equiv \sum_{m=2}^{\infty} \omega_{2m}\, \beta_{2m} \int_0^1 \Phi_c^0\, P_{2m}\, d\mu \Big/ \int_0^1 \Phi_c^0\, d\mu\,;$$

$$N_2^{\uparrow}(r) \equiv \sum_{m=2}^{\infty} \omega_{2m}\, \beta_{2m} \int_{-1}^0 \Phi_c^0\, P_{2m}\, d\mu \Big/ \int_{-1}^0 \Phi_c^0\, d\mu\,.$$

Using the moments of the first azimuthal harmonics of the scattering phase function related by the expression

$$\gamma_2(r, \mu) \equiv \int_{-1}^1 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' = \gamma_2^+(r, \mu) + \gamma_2^-(r, \mu) = \frac{4}{3}\,\omega_1(r)\sin\vartheta,$$

$$\gamma_2^+(r, \mu) \equiv \int_0^1 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \frac{2}{3}\,\omega_1\, \sin\vartheta + 2 \sum_{m=1}^{\infty} \frac{(2m-1)!}{(2m+1)!}\, \omega_{2m}\, \beta_{2m}^1\, P_{2m}^1\,,$$

$$\gamma_2^-(r, \mu) \equiv \int_{-1}^0 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \frac{2}{3}\,\omega_1\, \sin\vartheta - 2 \sum_{m=1}^{\infty} \frac{(2m-1)!}{(2m+1)!}\, \omega_{2m}\, \beta_{2m}^1\, P_{2m}^1\,,$$

we introduce the backscattering characteristics

$$\gamma_{2c}^{\downarrow}(r) \equiv \Gamma_{2c}^{\downarrow}/G_x^{\downarrow} = \frac{2}{3}\,\omega_1 - L_c^{\downarrow}\,;\; \gamma_{2c}^{\uparrow}(r) \equiv \Gamma_{2c}^{\uparrow}/G_x^{\uparrow} = \frac{2}{3}\,\omega_1 + L_c^{\uparrow}\,;$$

$$\gamma_{2s}^{\downarrow}(r) \equiv \Gamma_{2s}^{\downarrow}/G_y^{\downarrow} = \frac{2}{3}\,\omega_1 - L_s^{\downarrow}\,;\; \gamma_{2s}^{\uparrow}(r) \equiv \Gamma_{2s}^{\uparrow}/G_y^{\uparrow} = \frac{2}{3}\,\omega_1 + L_s^{\uparrow}\,.$$

The moments of radiation intensity depending on the scattering phase function are determined by the following formulas

$$\Gamma_{2c}^{\downarrow} \equiv \pi \int_0^1 \Phi_c^1(r, \mu)\, d\mu \int_{-1}^0 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \pi \int_0^1 \gamma_2^-(r, \mu)\, \Phi_c^1(r, \mu)\, d\mu\,;$$

$$\Gamma_{2c}^{\uparrow} \equiv \pi \int_{-1}^0 \Phi_c^1(r, \mu)\, d\mu \int_0^1 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \pi \int_{-1}^0 \gamma_2^+(r, \mu)\, \Phi_c^1(r, \mu)\, d\mu\,;$$

$$\Gamma_{2s}^{\downarrow} \equiv \pi \int_0^1 \Phi_s^1(r, \mu)\, d\mu \int_{-1}^0 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \pi \int_0^1 \gamma_2^-(r, \mu)\, \Phi_s^1(r, \mu)\, d\mu\,;$$

$$\Gamma_{2s}^{\uparrow} \equiv \pi \int_{-1}^0 \Phi_s^1(r, \mu)\, d\mu \int_0^1 \sin\vartheta'\, \gamma^1(r, \mu, \mu')\, d\mu' =$$

$$= \pi \int_{-1}^0 \gamma_2^+(r, \mu)\, \Phi_s^1(r, \mu)\, d\mu\,;$$

$$L_c^{\downarrow}(r) \equiv \sum_{m=1}^{\infty} \nu_{2m} \int_0^1 \Phi_c^1\, P_{2m}^1\, d\mu \Big/ \int_0^1 \Phi_c^1\, \sin\vartheta\, d\mu\,;$$

$$L_c^{\uparrow}(r) \equiv \sum_{m=1}^{\infty} \nu_{2m} \int_{-1}^0 \Phi_c^1\, P_{2m}^1\, d\mu \Big/ \int_{-1}^0 \Phi_c^1\, \sin\vartheta\, d\mu\,;$$

$$L_s^{\downarrow}(r) \equiv \sum_{m=1}^{\infty} \nu_{2m} \int_0^1 \Phi_s^1\, P_{2m}^1\, d\mu \Big/ \int_0^1 \Phi_s^1\, \sin\vartheta\, d\mu\,;$$

$$L_s^{\uparrow}(r) \equiv \sum_{m=1}^{\infty} \nu_{2m} \int_{-1}^0 \Phi_s^1\, P_{2m}^1\, d\mu \Big/ \int_{-1}^0 \Phi_s^1\, \sin\vartheta\, d\mu\,;$$

$$\nu_{2m} \equiv 2[(2m-1)!/(2m+1)!]\, \omega_{2m}(r)\, \beta_{2m}^1\,.$$

EQUATIONS FOR AZIMUTHAL HARMONICS

Using the expansions of Φ, Eqs. (4) and (5), and γ, Eq. (7), the collisional integral (2) is presented as a Fourier series

$$B(r, \mu, \varphi) = \frac{\sigma_s(r)}{2} \sum_{\kappa=0}^{\infty} \sum_{m=0}^{\kappa} \frac{1 + \delta_{m0}}{2\kappa + 1}\, \frac{(\kappa + m)!}{(\kappa - m)!}\, \gamma_\kappa^m(r) \times$$

$$\times [\Phi_{c\,\kappa}^{\,m}(r)\, C_\kappa^{\,m}(\mu, \varphi) + (1 - \delta_{m0})\, (1 - \delta_{\kappa 0})\, \Phi_{s\,\kappa}^{\,m}(r)\, S_\kappa^{\,m}(\mu, \varphi)] =$$

$$= \sum_{m=0}^{\infty} B_c^{\,m}(r, \mu)\, \cos m\varphi + B_s^{\,m}(r, \mu)\, \sin m\varphi \qquad (12)$$

with the azimuthal harmonics

$$B_c^{\,m}(r, \mu) = \frac{\sigma_s(r)}{4}\, \delta_m \int_{-1}^1 \Phi_c^{\,m}(r, \mu')\, \gamma^m(r, \mu, \mu')\, d\mu' =$$

$$= \frac{\sigma_s(r)}{2}\, \delta_m \sum_{\kappa=m}^{\infty} \frac{1}{2\kappa + 1}\, \frac{(\kappa + m)!}{(\kappa - m)!}\, \gamma_\kappa^m(r)\, \Phi_{c\,\kappa}^{\,m}(r)\, P_\kappa^{\,m}(\mu)\,; \qquad (13)$$

$$B_s^{\,m}(r, \mu) = (1 - \delta_{m0})\, \delta_m\, \frac{\sigma_s(r)}{4} \int_{-1}^1 \Phi_s^{\,m}(r, \mu')\, \gamma^m(r, \mu, \mu')\, d\mu' =$$

$$= (1 - \delta_{m0}) \frac{s_s(r)}{2} \delta_m \sum_{\kappa=m}^{\infty} \frac{1}{2\kappa + 1} \frac{(\kappa + m)!}{(\kappa - m)!}\, \gamma_\kappa^m(r)\, \Phi_{s\,\kappa}^{\,m}(r)\, P_\kappa^{\,m}(\mu). \qquad (14)$$

Substituting the expansions of Φ, Eq. (5), B, Eq. (12), and F, Eq. (10) into Eq. (1) and using the formulas for transforming the product of trigonometric functions into the sum we find the equality (see Eq. (27) in Ref. 9)

$$\mu \frac{\partial \Phi_c^0}{\partial z} + \sin\vartheta \left[\cos\varphi \frac{\partial \Phi_c^0}{\partial x} + \sin\varphi \frac{\partial \Phi_c^0}{\partial y} \right] + \sigma_t \Phi_c^0 +$$

$$+ \sum_{m=1}^{\infty} \cos m\varphi \left[\mu \frac{\partial \Phi_c^m}{\partial z} + \sigma_t \Phi_c^m \right] + \sum_{m=1}^{\infty} \sin m\varphi \left[\mu \frac{\partial \Phi_s^m}{\partial z} + \sigma_t \Phi_s^m \right] +$$

$$+ \frac{\sin\vartheta}{2} \sum_{m=1}^{\infty} \frac{\partial \Phi_c^m}{\partial x} [\cos(m-1)\varphi + \cos(m+1)\varphi] +$$

$$+ \frac{\partial \Phi_c^m}{\partial y} [\sin(m+1)\varphi - \sin(m-1)\varphi] +$$

$$+ \frac{\sin\vartheta}{2} \sum_{m=1}^{\infty} \frac{\partial \Phi_s^m}{\partial x} [\sin(m-1)\varphi + \sin(m+1)\varphi] +$$

$$+ \frac{\partial \Phi_s^m}{\partial y} [\cos(m-1)\varphi - \cos(m+1)\varphi] = B_c^0 + \sum_{m=1}^{\infty} B_c^m \cos m\varphi +$$

$$+ B_s^m \sin m\varphi + F_c^0 + \sum_{m=1}^{\infty} F_c^m \cos m\varphi + F_s^m \sin m\varphi \, . \qquad (15)$$

The equations for azimuthal harmonics can be constructed in two equivalent ways: either by integrating Eq. (15) over the azimuth $\varphi \in [0, 2\pi]$ with weights $\cos m\varphi$ and $\sin m\varphi$ using the condition of orthogonality of the trigonometric functions or by equalizing the expressions with the same trigonometric functions containing the azimuth. As a result, we obtain a system of equations which can be written in the generalized form,[9] as follows

$$\mu \frac{\partial \Phi_c^m}{\partial z} + \sigma_t \Phi_c^m + \frac{\sin\vartheta}{2} \left[\frac{\partial \Phi_c^{m+1}}{\partial x} + (1-\delta_{m0})(1+\delta_{m1}) \frac{\partial \Phi_c^{m-1}}{\partial x} + \frac{\partial \Phi_s^{m+1}}{\partial y} - (1-\delta_{m0})(1-\delta_{m1}) \frac{\partial \Phi_s^{m-1}}{\partial y} \right] = B_c^m + F_c^m \, , \qquad m \ge 0 \, ; \qquad (16)$$

$$\mu \frac{\partial \Phi_s^m}{\partial z} + \sigma_t \Phi_s^m + \frac{\sin\vartheta}{2} \left[\frac{\partial \Phi_s^{m+1}}{\partial x} + (1-\delta_{m1}) \frac{\partial \Phi_s^{m-1}}{\partial x} - \frac{\partial \Phi_c^{m+1}}{\partial y} + (1+\delta_{m1}) \frac{\partial \Phi_c^{m-1}}{\partial y} \right] = B_s^m + F_s^m \, , \qquad m \ge 1 \, ; \qquad (17)$$

In contrast to a one–dimension plane problem[5] the system of equations (16)–(17) for the azimuthal harmonics of the three–dimensional problem (1) cannot be splitted.

EQUATIONS FOR SPHERICAL HARMONICS

The equations for spherical harmonics of radiation intensity in a three–dimensional plane layer can be constructed in two ways.

In the first method the expansions of Φ, Eq. (4), B, Eq. (12), and F, Eq. (10), over spherical functions are substituted into Eq. (1). The equality obtained (see Eq. (37) in Ref. 9) is first multiplied by $C_j^i(\mu, \varphi)$, $0 \le i \le j$, and integrated over the sphere Ω. Then the equality is multiplied by $S_j^i(\mu, \varphi)$, $1 \le i \le j$, and integrated over the sphere Ω. Using explicit expressions of integrals with spherical functions (see Appendix in Ref. 9) we find the system of equations for spherical harmonics[9]:

$$\delta_m \left\{ (1-\delta_{j0})(1-\delta_{jm}) a_j^m \frac{\partial \Phi_{c,j-1}^m}{\partial z} + b_j^m \frac{\partial \Phi_{c,j+1}^m}{\partial z} + \left[\sigma_t - \sigma_s h_j^m \gamma_j^m \right] \Phi_{cj}^m \right\} +$$

$$+ \frac{1}{2} \left\{ \delta_m \left[c_j^m \frac{\partial \Phi_{c,j+1}^{m+1}}{\partial x} - (1-\delta_{j0})(1-\delta_{j1})(1-\delta_{jm})(1-\delta_{j\,m+1}) d_j^m \frac{\partial \Phi_{c,j-1}^{m+1}}{\partial x} \right] + (1-\delta_{m0})(1-\delta_{j0})(1+\delta_{m1}) \times \right.$$

$$\left. \times \left[g_{j-1} \frac{\partial \Phi_{c,j-1}^{m-1}}{\partial x} - g_{j+1} \frac{\partial \Phi_{c,j+1}^{m-1}}{\partial x} \right] \right\} + \frac{1}{2} \left\{ \delta_m \left[c_j^m \frac{\partial \Phi_{s,j+1}^{m+1}}{\partial y} - (1-\delta_{j0})(1-\delta_{j1})(1-\delta_{jm})(1-\delta_{j\,m+1}) d_j^m \frac{\partial \Phi_{s,j-1}^{m+1}}{\partial y} \right] - \right.$$

$$\left. - (1-\delta_{m0})(1-\delta_{m1})(1-\delta_{j0})(1-\delta_{j1}) \left[g_{j-1} \frac{\partial \Phi_{s,j-1}^{m-1}}{\partial y} - g_{j+1} \frac{\partial \Phi_{s,j+1}^{m-1}}{\partial y} \right] \right\} = \delta_m F_{cj}^m, \qquad 0 \le m \le j \, ; \qquad (18)$$

$$(1-\delta_{j1})(1-\delta_{jm}) a_j^m \frac{\partial \Phi_{s,j-1}^m}{\partial z} + b_j^m \frac{\partial \Phi_{s,j+1}^m}{\partial z} + \left[\sigma_t - \sigma_s h_j^m \gamma_j^m \right] \Phi_{sj}^m +$$

$$+ \frac{1}{2} \left\{ c_j^m \frac{\partial \Phi_{s,j+1}^{m+1}}{\partial x} - (1-\delta_{j1})(1-\delta_{jm})(1-\delta_{j\,m+1}) d_j^m \frac{\partial \Phi_{s,j-1}^{m+1}}{\partial x} + (1-\delta_{m1}) \left[(1-\delta_{j1}) g_{j-1} \frac{\partial \Phi_{s,j-1}^{m-1}}{\partial x} - g_{j+1} \frac{\partial \Phi_{s,j+1}^{m-1}}{\partial x} \right] \right\} +$$

$$+ \frac{1}{2} \left\{ (1+\delta_{m1}) \left[g_{j-1} \frac{\partial \Phi_{c,j-1}^{m-1}}{\partial y} - g_{j+1} \frac{\partial \Phi_{c,j+1}^{m-1}}{\partial y} \right] - c_j^m \frac{\partial F_{c,j+1}^{m+1}}{\partial y} + (1-\delta_{j1})(1-\delta_{jm})(1-\delta_{j\,m+1}) d_j^m \frac{\partial \Phi_{c,j-1}^{m+1}}{\partial y} \right\} = F_{sj}^m \, , \qquad 1 \le m \le j \, , \qquad (19)$$

where the following designations for the coefficients are introduced:

$$a_j^m = \frac{j-m}{2j-1} \, ; \; b_j^m = \frac{j+m+1}{2j+3} \, ; \; g_j = \frac{1}{2j+1} \, ;$$

$$h_j^m = \frac{1+\delta_{m0}}{2\,(2j+1)} \frac{(j+m)!}{(j-m)!} \, ; \; c_j^m = \frac{(j+m+1)(j+m+2)}{(2j+3)} \, ;$$

$$d_j^m = \frac{(j-m-1)(j-m)}{(2j-1)} \, .$$

According to the second method the expansions of F_c^m, F_s^m, Eq. (6), B_c^m, Eq. (13), B_s^m, Eq. (14), F_c^m, F_s^m, Eq. (11), over the associated Legendre functions are substituted into Eqs. (16)–(17) for the azimuthal harmonics of radiation intensity. Using the recurrence formulas (see Eqs. (45)–(47), Ref. 9) we eliminate the multipliers μ and $\sin\vartheta$ and simultaneously reduce the associated Legendre functions to a single upper index equal to m. The obtained equations (see Eqs. (48) and (49), Ref. 9) are integrated over μ on the segment $[-1, 1]$ with the weight $P_j^m(\mu)$ using the properties of orthogonality of the associated Legendre functions with equal upper indices. Finally we obtain Eqs. (18) and (19).

The system of equations (18)–(19) is a system of equations for spherical harmonics of the solution of the problem (1) in the most general form. Different modifications, taking into account azimuthal symmetry, inhomogeneity along one or two horizontal axes, x and y, as well as any approximations of lower orders (for example, P_1– or P_2–approximation), can be derived from this system.

EXACT MODELS FOR CALCULATING RADIATION DENSITIES AND FLUXES

Let us formulate exact, closed mathematical models[11] describing spatial distributions of density $n(r)$ and fluxes $J(r)$, $G_x(r)$, and $G_y(r)$ of radiation in three–dimensional scattering and absorbing layers based on four exact equations of the system (16)–(17) for azimuthal harmonics of solution of the general boundary value problem of the transfer theory (1).

Let us integrate Eq. (16) over μ in the interval $[-1, 1]$ with unit weight for $m = 0$

$$\mu \frac{\partial \Phi_c^0}{\partial z} + \sigma_t \Phi_c^0 + \frac{\sin\vartheta}{2}\left[\frac{\partial \Phi_c^1}{\partial x} + \frac{\partial \Phi_s^1}{\partial y}\right] = B_c^0 + F_c^0 \tag{20}$$

and, as a result, we obtain the first exact equation

$$\frac{\partial J}{\partial z} + (\sigma_t - \sigma_s \omega_0)\, n + \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} = 4\pi F_{c0}^0 . \tag{21}$$

Let us now integrate the same equation (16) over μ in the interval $[-1, 1]$ with the weight μ and derive the second exact equation

$$\frac{\partial [D\, n]}{\partial z} + \left[\sigma_t - \frac{\sigma_s \omega_1}{3}\right] J + \frac{\partial [\mu_x G_x]}{\partial x} + \frac{\partial [\mu_y G_y]}{\partial y} = \frac{4\pi}{3} F_{c1}^0 \tag{22}$$

with the radiation parameters $D(r)$, $\mu_x(r)$, and $\mu_y(r)$. Equation (16) with $m = 1$

$$\mu \frac{\partial \Phi_c^1}{\partial z} + \sigma_t \Phi_c^1 + \sin\vartheta \left\{\frac{\partial \Phi_c^0}{\partial x} + \frac{1}{2}\left[\frac{\partial \Phi_c^2}{\partial x} + \frac{\partial \Phi_s^2}{\partial y}\right]\right\} = B_c^1 + F_c^1 \tag{23}$$

is integrated over m ∈ $[-1, 1]$ with the weight $\sin\vartheta$ and the third exact equation

$$\frac{\partial [\mu_x G_x]}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s \omega_1\right) G_x +$$
$$+ \frac{1}{2}\left\{\frac{\partial [(D_\perp + D_{\perp c2})\, n]}{\partial x} + \frac{\partial [D_{\perp s2}\, n]}{\partial y}\right\} = \frac{4\pi}{3} F_{c1}^1 \tag{24}$$

with the radiation parameters μ_x, $D_\perp$, $D_{\perp c2}$, and $D_{\perp s2}$ is derived. Similar actions are undertaken with Eq. (17) with $m = 1$

$$\mu \frac{\partial \Phi_s^1}{\partial z} + \sigma_t \Phi_s^1 + \sin\vartheta \left\{\frac{\partial \Phi_c^0}{\partial y} + \frac{1}{2}\left[\frac{\partial \Phi_s^2}{\partial x} - \frac{\partial \Phi_c^2}{\partial y}\right]\right\} =$$
$$= B_s^1 + F_s^1 \tag{25}$$

and the fourth exact equation

$$\frac{\partial [\mu_y G_y]}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s \omega_1\right) G_y +$$
$$+ \frac{1}{2}\left\{\frac{\partial [D_{\perp s2}\, n]}{\partial x} + \frac{\partial [(D_\perp - D_{\perp c2})\, n]}{\partial y}\right\} = \frac{4\pi}{3} F_{s1}^1 \tag{26}$$

with the radiation parameters μ_y, $D_\perp$, $D_{\perp c2}$, and $D_{\perp s2}$ is obtained.

If the expressions for radiation characteristics in terms of spherical harmonics are substituted into Eq. (21), we derive the exact equation (18) of the method of spherical harmonics with indices $m = 0$, $j = 0$:

$$\frac{1}{3}\frac{\partial \Phi_{c1}^0}{\partial z} + \left(\sigma_t - \sigma_s \gamma_0^0\right) \Phi_{c0}^0 + \frac{1}{3}\frac{\partial \Phi_{c1}^1}{\partial x} + \frac{1}{3}\frac{\partial \Phi_{s1}^1}{\partial y} = F_c^0 .$$

Equation (22) with the vertical diffusion coefficient D represented in terms of spherical harmonics is equivalent to the exact equation (18) with indices $m = 0$, $j = 1$:

$$\frac{2}{5}\frac{\partial \Phi_{c2}^0}{\partial z} + \frac{\partial \Phi_{c0}^0}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s \gamma_1^0\right) \Phi_{c1}^0 + \frac{3}{5}\frac{\partial \Phi_{c2}^1}{\partial x} + \frac{3}{5}\frac{\partial \Phi_{s2}^1}{\partial y} = F_{c1}^0$$

If in Eq. (24) we express radiation characteristics μ_x, G_x, n, $D_\perp$, $D_{\perp c2}$, and $D_{\perp s2}$ in terms of spherical harmonics, we obtain the equation

$$\frac{3}{5}\frac{\partial \Phi_{c2}^1}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s \omega_1\right) \Phi_{c1}^1 + \frac{\partial \Phi_{c0}^0}{\partial x} - \frac{1}{5}\frac{\partial \Phi_{c2}^0}{\partial x} +$$
$$+ \frac{6}{5}\frac{\partial \Phi_{c2}^2}{\partial x} + \frac{6}{5}\frac{\partial \Phi_{s2}^2}{\partial y} = F_{c1}^1 ,$$

which coincides with Eq. (18) of the method of spherical harmonics with $m = 1$, $j = 1$. If in Eq. (26) we substitute the expressions of radiation characteristics μ_y, G_y, n, $D_\perp$, $D_{\perp c2}$, and $D_{\perp s2}$ in terms of spherical harmonics, then Eq. (26) becomes equivalent to Eq. (19) of the method of spherical harmonics with $m = 1$, $j = 1$:

$$\frac{3}{5}\frac{\partial \Phi_{s2}^1}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s \omega_1\right) \Phi_{s1}^1 + \frac{\partial \Phi_{c0}^0}{\partial y} - \frac{1}{5}\frac{\partial \Phi_{c2}^0}{\partial y} +$$
$$+ \frac{6}{5}\frac{\partial \Phi_{s2}^2}{\partial x} - \frac{6}{5}\frac{\partial \Phi_{c2}^2}{\partial y} = F_{s1}^1 .$$

The system of four equations (21), (22), (24), and (26) with the radiation parameters μ_x, μ_y, D, $D_\perp$, $D_{\perp c2}$, and $D_{\perp s2}$ is the exact closed mathematical model for calculating the density n and fluxes J, G_x, and G_y of radiation in a homogeneous three–dimensional layer, which has been constructed using the method of spherical harmonics.

The spherical (integral over all angles) radiation characteristics are related exactly by the following relations:

$$J = \bar{\mu}\, n , \qquad G_x = c_x J = s_x n , \qquad G_y = c_y J = s_y n , \tag{27}$$

which involve the following radiation parameters: $\bar{\mu}$, c_x, c_y are the mean cosines along the axes z, x, and y, respectively, and

s_x and s_y are the mean sines along the axes x and y, respectively. Using relations (27) in the system of equations (21), (22), (24), and (26) it is possible to change a set of unknown functions and nonlinear parameters of the problem taking into account the requirements of a particular problem.

From Eq. (21) we derive the representation of radiation density through the fluxes ($\sigma_a \equiv \sigma_t - \sigma_s$):

$$n = -\frac{1}{\sigma_a}\left[\frac{\partial J}{\partial z} + \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} - 4\pi F^0_{c0}\right], \tag{28}$$

with the help of which Eqs. (22), (24), and (26) are reduced to the system of three differential equations of the second order with mixed derivatives for determining the fluxes J, G_x, and G_y:

$$\frac{\partial}{\partial z}\frac{D}{\sigma_a}\frac{\partial J}{\partial z} + \frac{\partial}{\partial z}\frac{D}{\sigma_a}\frac{\partial G_x}{\partial x} + \frac{\partial}{\partial z}\frac{D}{\sigma_a}\frac{\partial G_y}{\partial y} - \frac{\partial[\mu_x G_x]}{\partial x} - \frac{\partial[\mu_y G_y]}{\partial y} -$$

$$- \left(\sigma_t - \frac{1}{3}\sigma_s\,\omega_1\right) J = -4\pi\left[\frac{1}{3}F^0_{c1} - \frac{\partial}{\partial z}\frac{D}{\sigma_a}F^0_{c0}\right]; \tag{29}$$

$$\frac{1}{2}\frac{\partial}{\partial x}\frac{D_\perp + D_{\perp c2}}{\sigma_a}\frac{\partial J}{\partial z} + \frac{1}{2}\frac{\partial}{\partial x}\frac{D_\perp + D_{\perp c2}}{\sigma_a}\frac{\partial G_x}{\partial x} +$$

$$+ \frac{1}{2}\frac{\partial}{\partial x}\frac{D_\perp + D_{\perp c2}}{\sigma_a}\frac{\partial G_y}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial J}{\partial z} + \frac{1}{2}\frac{\partial}{\partial y}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial G_x}{\partial x} +$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial G_y}{\partial y} - \frac{\partial[\mu_x G_x]}{\partial z} - \left(\sigma_t - \frac{1}{3}\sigma_s\,\omega_1\right) G_x =$$

$$= -4\pi\left[\frac{1}{3}F^1_{c1} - \frac{1}{2}\frac{\partial}{\partial x}\frac{D_\perp + D_{\perp c2}}{s_a}F^0_{c0} - \frac{1}{2}\frac{\partial}{\partial y}\frac{D_{\perp s2}}{\sigma_a}F^0_{c0}\right]; \tag{30}$$

$$\frac{1}{2}\frac{\partial}{\partial x}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial J}{\partial z} + \frac{1}{2}\frac{\partial}{\partial x}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial G_x}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\frac{D_{\perp s2}}{\sigma_a}\frac{\partial G_y}{\partial y} +$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\frac{D_\perp - D_{\perp c2}}{\sigma_a}\frac{\partial J}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y}\frac{D_\perp - D_{\perp c2}}{\sigma_a}\frac{\partial G_x}{\partial x} +$$

$$+ \frac{1}{2}\frac{\partial}{\partial y}\frac{D_\perp - D_{\perp c2}}{\sigma_a}\frac{\partial G_y}{\partial y} - \frac{\partial[\mu_y G_y]}{\partial z} - \left(\sigma_t - \frac{1}{3}\sigma_s\,\omega_1\right) G_y =$$

$$= -4\pi\left[\frac{1}{3}F^1_{s1} - \frac{1}{2}\frac{\partial}{\partial x}\frac{D_{\perp s2}}{\sigma_a}F^0_{c0} - \frac{1}{2}\frac{\partial}{\partial y}\frac{D_\perp - D_{\perp c2}}{\sigma_a}F^0_{c0}\right]. \tag{31}$$

The system of equations (29)–(31) of the type of diffusion equations is the exact nonlinear mathematical model for calculating the radiation fluxes.

EXACT MODELS FOR CALCULATING HEMISPHERICAL DENSITIES AND VERTICAL AND HORIZONTAL FLUXES OF RADIATION

The models[11] are constructed based on three exact equations (20), (23), and (25) for azimuthal harmonics of radiation intensity when azimuthal harmonics of the collisional integral (13) and (14) and the source (11) are presented as expansions over the associated Legendre functions.

First, it should be noted that the system of equations (20), (23), and (25) is not closed since the system of equations (16)–(17) for azimuthal harmonics of radiation intensity is infinite. Second, the expansions of hemispherical radiation characteristics integral over angles, $n^\downarrow$, $n^\uparrow$, $J^\downarrow$, $J^\uparrow$, $G^\downarrow_x$, $G^\uparrow_x$, $G^\downarrow_y$, and $G^\uparrow_y$, over the associated Legendre functions are sums of infinite series. Third, the densities $n^\downarrow$ and $n^\uparrow$ and the vertical fluxes $J^\downarrow$ and $J^\uparrow$ are completely determined through the zeroth azimuthal harmonic F^0_c, the horizontal fluxes along the axis x $G^\downarrow_x$, $G^\uparrow_x$ are determined by the first azimuthal harmonics F^1_c, and the horizontal fluxes along the axis y $G^\downarrow_y$, $G^\uparrow_y$ are determined by the azimuthal harmonics F^1_s. Fourth, the radiation characteristics are related to each other through a set of exact and approximate relations which enable one to construct different calculational models. Fifth, the problem on making the system of exact equations for calculating the aforementioned hemispherical radiation characteristics closed is uncertain, since one has to introduce nonlinear parameters depending on radiation characteristics. These parameters should describe the radiation transfer in a medium and have clear physical meaning.

Let us integrate Eq. (29) over μ on the segments [0, 1] and [–1, 0] with the weight equal to unity. Using the definitions of radiation characteristics and radiation parameters in terms of azimuthal harmonics we calculate the integrals in an explicit form and obtain the first pair of exact equations:

$$\frac{\partial J^\downarrow}{\partial z} + \left[\sigma_t - \sigma_s + \frac{\sigma_s}{2}\gamma^\downarrow_0\right] n^\downarrow - \frac{\sigma_s}{2}\gamma^\uparrow_0\, n^\uparrow + \frac{\partial G^\downarrow_x}{\partial x} + \frac{\partial G^\downarrow_y}{\partial y} = 2\pi Q^\downarrow_0 \tag{32}$$

$$\frac{\partial J^{\text{-}}}{\partial z} + \left[\sigma_t - \sigma_s + \frac{\sigma_s}{2}\gamma^\uparrow_0\right] n^\uparrow - \frac{\sigma_s}{2}\gamma^\downarrow_0\, n^\downarrow + \frac{\partial G^\uparrow_x}{\partial x} + \frac{\partial G^\uparrow_y}{\partial y} = 2\pi Q^\uparrow_0 \tag{33}$$

with the parameters $\gamma^\downarrow_0$ and $\gamma^\uparrow_0$ and the sources

$$Q^\downarrow_0 \equiv \int_0^1 F^0_c(r, \mu)\, d\mu = F^0_{c0} + \frac{1}{2}F^0_{c1} + \sum_{m=1}^{\infty} t^0_{2m+1}\, F^0_{c,\,2m+1}\,;$$

$$Q^\uparrow_0 \equiv \int_{-1}^0 F^0_c(r, \mu)\, d\mu = F^0_{c0} - \frac{1}{2}F^0_{c1} - \sum_{m=1}^{\infty} t^0_{2m+1}\, F^0_{c,\,2m+1}\,.$$

Let us now integrate Eq. (20) over μ on the segments [0, 1] and [–1, 0] with the weight μ. After some simple transformations we find the second pair of exact equations

$$\frac{\partial[D^\downarrow n^\downarrow]}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s\,\omega_1\right) J^\downarrow + \frac{\sigma_s}{2}\gamma^\downarrow_1\, n^\downarrow - \frac{\sigma_s}{2}\gamma^\uparrow_1\, n^\uparrow +$$

$$+ \frac{\partial[\mu^\downarrow_x G^\downarrow_x]}{\partial x} + \frac{\partial[\mu^\downarrow_y G^\downarrow_y]}{\partial y} = 2\pi Q^\downarrow_1\,; \tag{34}$$

$$\frac{\partial[D^\uparrow n^\uparrow]}{\partial z} + \left(\sigma_t - \frac{1}{3}\sigma_s\,\omega_1\right) J^\uparrow - \frac{\sigma_s}{2}\gamma^\downarrow_1\, n^\downarrow + \frac{\sigma_s}{2}\gamma^\uparrow_1\, n^\uparrow +$$

$$+ \frac{\partial[\mu^\uparrow_x G^\uparrow_x]}{\partial x} + \frac{\partial[\mu^\uparrow_y G^\uparrow_y]}{\partial y} = 2\pi Q^\uparrow_1 \tag{35}$$

with the parameters and the sources

$$Q^\downarrow_1 \equiv \int_0^1 \mu\, F^0_c(r, \mu)\, d\mu = \frac{1}{2}F^0_{c0} + \frac{1}{3}F^0_{c1} + \sum_{m=1}^{\infty} \beta_{2m}\, F^0_{c,\,2m}\,;$$

$$Q^\uparrow_1 \equiv \int_{-1}^0 \mu\, F^0_c(r, \mu)\, d\mu = -\frac{1}{2}F^0_{c0} + \frac{1}{3}F^0_{c1} - \sum_{m=1}^{\infty} \beta_{2m}\, F^0_{c,\,2m}\,.$$

Then we integrate equation (23) over μ on the segments [0, 1] and [−1, 0] with the weight $\sin\vartheta$ and obtain the third pair of exact equations

$$\frac{\partial[\mu_x^{\downarrow} G_x^{\downarrow}]}{\partial z} + \left\{\sigma_t - \frac{\sigma_s}{2}\left[\frac{4}{3}\omega_1 - \gamma_{2c}^{\downarrow}\right]\right\} G_x^{\downarrow} - \frac{\sigma_s}{2}\gamma_{2c}^{\uparrow} G_x^{\uparrow} +$$

$$+ \frac{1}{2}\frac{\partial[(D_{\perp}^{\downarrow} + D_{\perp c2}^{\downarrow})\, n^{\downarrow}]}{\partial x} + \frac{1}{2}\frac{\partial[D_{\perp s2}^{\downarrow}\, n^{\downarrow}]}{\partial y} = \pi\, Q_{2c}^{\downarrow}\ ; \qquad (36)$$

$$\frac{\partial[\mu_x^{\uparrow} G_x^{\uparrow}]}{\partial z} + \left\{\sigma_t - \frac{\sigma_s}{2}\left[\frac{4}{3}\omega_1 - \gamma_{2c}^{\uparrow}\right]\right\} G_x^{\uparrow} - \frac{\sigma_s}{2}\gamma_{2c}^{\downarrow} G_x^{\downarrow} +$$

$$+ \frac{1}{2}\frac{\partial[(D_{\perp}^{\uparrow} + D_{\perp c2}^{\uparrow})\, n^{\uparrow}]}{\partial x} + \frac{1}{2}\frac{\partial[D_{\perp s2}^{\uparrow}\, n^{\uparrow}]}{\partial y} = \pi\, Q_{2c}^{\uparrow} \qquad (37)$$

with the parameters and the sources

$$Q_{2c}^{\downarrow}(r) \equiv \int_0^1 \sin\vartheta\, F_c^1(r, \mu)\, d\mu = \frac{2}{3} F_{c1}^1 + \sum_{m=1}^{\infty} \beta_{2m} F_{c,\,2m}^1\ ;$$

$$Q_{2c}^{\uparrow}(r) \equiv \int_{-1}^0 \sin\vartheta\, F_c^1(r, \mu)\, d\mu = \frac{2}{3} F_{c1}^1 - \sum_{m=1}^{\infty} \beta_{2m} F_{c,\,2m}^1\ .$$

Let Eq. (25) be integrated over μ on the segments [0, 1] and [−1, 0] with the weight $\sin\vartheta$. After some simple transformations we obtain the fourth pair of exact equations:

$$\frac{\partial[\mu_y^{\downarrow} G_y^{\downarrow}]}{\partial z} + \left\{\sigma_t - \frac{\sigma_s}{2}\left[\frac{4}{3}\omega_1 - \gamma_{2s}^{\downarrow}\right]\right\} G_y^{\downarrow} - \frac{\sigma_s}{2}\gamma_{2s}^{\uparrow} G_y^{\uparrow} +$$

$$+ \frac{1}{2}\frac{\partial[D_{\perp s2}^{\downarrow}\, n^{\downarrow}]}{\partial x} + \frac{1}{2}\frac{\partial[(D_{\perp r}^{\downarrow} - D_{\perp c2}^{\downarrow})\, n^{\downarrow}]}{\partial y} = \pi\, Q_{2s}^{\downarrow}\ ; \qquad (38)$$

$$\frac{\partial[\mu_y^{\uparrow} G_y^{\uparrow}]}{\partial z} + \left\{\sigma_t - \frac{\sigma_s}{2}\left[\frac{4}{3}\omega_1 - \gamma_{2s}^{\uparrow}\right]\right\} G_y^{\uparrow} - \frac{\sigma_s}{2}\gamma_{2s}^{\downarrow} G_y^{\downarrow} +$$

$$+ \frac{1}{2}\frac{\partial[D_{\perp s2}^{\uparrow}\, n^{-}]}{\partial x} + \frac{1}{2}\frac{\partial[(D_{r}^{\uparrow} - D_{\perp c2}^{\uparrow})\, n^{-}]}{\partial y} = \pi\, Q_{2s}^{\uparrow} \qquad (39)$$

with the parameters and the sources

$$Q_{2s}^{\downarrow}(r) \equiv \int_0^1 \sin\vartheta\, F_s^1(r, \mu)\, d\mu = \frac{2}{3} F_{s1}^1 + \sum_{m=1}^{\infty} \beta_{2m}^1 F_{s,\,2m}^1\ ;$$

$$Q_{2s}^{\uparrow}(r) \equiv \int_{-1}^0 \sin\vartheta\, F_s^1(r, \mu)\, d\mu = \frac{2}{3} F_{s1}^1 - \sum_{m=1}^{\infty} \beta_{2m}^1 F_{s,\,2m}^1\ .$$

Using the exact relations between hemispherical radiation characteristics

$$J^{\downarrow} = \mu^{\downarrow} n^{\downarrow}\ ;\quad J^{\uparrow} = \mu^{\uparrow} n^{\uparrow}\ ;$$

$$G_x^{\downarrow} = c_x^{\downarrow} J^{\downarrow} = (\mu^{\downarrow} c_x^{\downarrow})\, n^{\downarrow} = s_x^{\downarrow} n^{\downarrow}\ ;\ G_y^{\downarrow} = c_y^{\downarrow} J^{\downarrow} = (\mu^{\downarrow} c_y^{\downarrow})\, n^{\downarrow} = s_y^{\downarrow} n^{\downarrow};$$

$$G_x^{\uparrow} = c_x^{\uparrow} J^{\uparrow} = (\mu^{\uparrow} c_x^{\uparrow})\, n^{\uparrow} = s_x^{\uparrow} n^{\uparrow}\ ;\ G_y^{\uparrow} = c_y^{\uparrow} J^{\uparrow} = (\mu^{\uparrow} c_y^{\uparrow})\, n^{\uparrow} = s_y^{\uparrow} n^{\uparrow}$$

and the exact expressions for backscattering characteristics expressed in terms of moments it is possible to obtain different representations of the pairs of equations (32)–(33), (34)–(35), (36)–(37), and (38)–(39) with different sets of sought functions and nonlinear coefficients, i.e. parameters of mathematical models.

APPENDIX

To formulate mathematical models for calculating hemispherical radiation characteristics and radiation parameters as well as boundary conditions, we obtained the explicit expressions of integrals in terms of the product of the associated Legendre functions along the segment [0, 1].

If $n \ne \kappa$ and κ is odd and n is even, i.e. $\kappa + n$ and $|\kappa - n|$ are odd, then

$$a_{n\kappa}^m \equiv \int_0^1 P_n^m(\mu)\, P_\kappa^m(\mu)\, d\mu = (-1)^{\frac{n+\kappa+1}{2}} \times$$

$$\times \frac{(\kappa + m)!\ (n + m)!}{2^{n+\kappa-1}\ (n - \kappa)\ (n + k + 1)\ b_{n\kappa}^m}\ , \qquad (A)$$

$$b_{n\kappa}^m = \begin{cases} \left[\left(\dfrac{n}{2}\right)!\left(\dfrac{\kappa-1}{2}\right)!\right]^2, & \text{if } m = 0; \\ \left(\dfrac{n+m}{2}\right)!\left(\dfrac{n-m}{2}\right)!\left(\dfrac{\kappa+m-1}{2}\right)!\left(\dfrac{\kappa-m-1}{2}\right)!\ , & \\ \text{if } m \text{ is even}; & \\ \left(\dfrac{\kappa+m}{2}\right)!\left(\dfrac{\kappa-m}{2}\right)!\left(\dfrac{n+m-1}{2}\right)!\left(\dfrac{n-m-1}{2}\right)!, & \\ \text{if } m \text{ is odd}\ . & \end{cases}$$

If $n \ne \kappa$, but k and n are even or κ and n are odd, i.e. $\kappa + n$ and $|\kappa - n|$ are even, then

$$\int_0^1 P_n^m(\mu)\, P_\kappa^m(\mu)\, d\mu = 0\ .$$

If $n = \kappa$, then Eq. (A) cannot be used. One must take for

$$a_{nn}^m = \int_0^1 [P_n^m(\mu)]^2\, d\mu = \frac{1}{2n+1}\frac{(n+m)!}{(n-m)!}\ , \qquad m \ge 1\ ; \qquad \text{and}$$

$$a_{nn}^0 = \int_0^1 [P_n(\mu)]^2\, d\mu = \frac{1}{2n+1}\ ,\ \text{ if } m = 0\ .$$

If $m = 0$, $n = 0$, and κ is odd, then

$$a_{0\kappa}^0 = \int_0^1 P_k(\mu)\, d\mu = \frac{(-1)^{\frac{\kappa+3}{2}}\ (\kappa - 1)!}{2^{\kappa-1}\ (\kappa + 1)\left[\left(\dfrac{\kappa-1}{2}\right)!\right]^2}\ ;$$

and for even κ values

$$\int_0^1 P_\kappa(\mu)\, d\mu = 0\ .$$

When $m = 0$ one can obtain the alternative expression for the integral

$$a_{2n,\,2\kappa+1}^0 = \int_0^1 P_{2n}(\mu)\, P_{2\kappa+1}(\mu) d\mu =$$

$$= \frac{(-1)^{n+\kappa+1}\ (2\kappa + 1)!!\ (2n - 1)!!}{2^{n+\kappa+1}\ (2n - 2\kappa - 1)\ (n + \kappa + 1)\ n!\ \kappa!}\ ,$$

if the equalities

$$(2n)! = 2^n\,(2n-1)!!\,n!\ ,\ (2\kappa+1)! = 2^\kappa\,(2\kappa+1)!!\,\kappa!$$

are used.

Let us write the generalized expression for the explicit integral value

$$\int\limits_0^1 P_1^m(\mu)\,P_r^m(\mu)\,\mathrm{d}\mu = \delta_{1r}\,a_{11}^m + \delta_{1,\,2n}\,\delta_{r,\,2\kappa+1}\,a_{2n,\,2\kappa+1}^m + $$
$$+\,\delta_{r,\,2n}\,\delta_{1,\,2\kappa+1}\,a_{2n,\,2\kappa+1}^m\ .$$

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