LIGHT SCATTERING BY A VOLUME SCATTERING ELEMENT IN THE CASE OF A FOCUSED INCIDENT BEAM

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The problem of light scattering by a volume scattering element in the case of a focused incident beam has been solved using an expansion of a scattering matrix in generalized spherical functions. Some particular cases are considered. The effect of incident beam geometry on spatial distribution of scattered radiation (scattering phase function) is illustrated.

In theoretical studies in the field of optics of dispersed media the assumption that the incident radiation represents a plane electromagnetic wave or a parallel beam is not always justified. In optical experiment one should take into account the geometry[1–2] and structure[3–4] of incident radiation.

The present paper is concerned with scattering of focused or divergent beam by a volume scattering element and with the effect of the incident beam geometry on the amount of radiation scattered at different angles.

The volume scattering element contains randomly oriented particles having a plane of symmetry and/or particles and their mirror images in equal proportion with random orientation.

Let the incident radiation be focused or divergent beam in the form of a cone whose directrix coincides with the Z axis and degree of convergence or divergence is determined by an angle (ϑ_0') between the directrix and generatrix of this cone.

The particles in the volume scattering element are assumed to be randomly positioned. For this reason the beams scattered by individual particles are incoherent. It allows one to apply the principle of additivity of the Stokes parameters. The result of interaction between the incident beam and the given volume is the sum of the results of interaction between each local beam (parallel by convention) and this volume.

On the basis of these assumptions the solution for the focused and divergent beam with equal degree of divergence or convergence and identical structure (intensity and polarization) has the same form.

Two representations of the electric field strength and the corresponding systems of the Stokes parameters and scattering matrices are used in this paper. In the CP–representation the components of the electric field strength can be written in the form[5]

$$\begin{bmatrix} E_{+1} \\ E_{-1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \qquad (1)$$

where E_1 and E_2 are the parallel and perpendicular components in the LP–representation[6] referred to the reference plane.

The systems of the Stokes parameters for incident and scattered radiation in the LP–representation[6] are defined as

$$I = E_1 E_1^* + E_2 E_2^* , \quad Q = E_1 E_1^* - E_2 E_2^* ,$$
$$U = E_1 E_2^* + E_2 E_1^*, \qquad V = i\,(E_2 E_1^* - E_1 E_2^*) ; \qquad (2)$$

while in the CP–representation[5]

$$I_2 = E_{+1} E_{-1}^* = \frac{1}{2}(Q - iU) , \quad I_0 = E_{+1} E_{+1}^* = \frac{1}{2}(I - V) ,$$
$$I_{-0} = E_{-1} E_{-1}^* = \frac{1}{2}(I + V) , \quad I_{-2} = E_{-1} E_{+1}^* = \frac{1}{2}(Q + iU) ,$$
$$\mathbf{I}^L = (I, Q, U, V)^{\mathrm{T}} , \qquad \mathbf{I}^C = (I_2, I_0, I_{-0}, I_{-2})^{\mathrm{T}} , \qquad (3)$$

where the asterisk denotes complex conjugation and T – transposition.

The transformation of the Stokes parameters I^L into I^C can be written in the form

$$\mathbf{I}^C = \mathbf{A}\,\mathbf{I}^L , \qquad (4)$$

where

$$\mathbf{A} = \frac{1}{2} \begin{bmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{bmatrix} . \qquad (5)$$

The inverse transformation has the form

$$\mathbf{I}^L = \mathbf{A}^{-1}\,\mathbf{I}^C , \qquad (6)$$

where

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ i & 0 & 0 & -i \\ 0 & -1 & 1 & 0 \end{bmatrix} \qquad (7)$$

(here the superscript –1 denotes the inverse matrix).

Let us consider the system of coordinates (Fig. 1) in which the directions of scattering and propagation of a local beam are specified by the spherical angels (ϑ, φ) and (ϑ', φ'), respectively.

The transformation of the Stokes parameters of incident radiation into the Stokes parameters of scattered radiation caused by light scattering by the volume scattering element depends on the scattering angle θ and is given as[5–7]

$$\mathbf{I}^{C,\,L}_{\rm sc} = r^{-2}\,\mathbf{Z}^{C,\,L}(\theta)\,\mathbf{I}^{C,\,L}_{\rm i}\ , \tag{8}$$

where r is the distance to the observation point, $\mathbf{I}^{C,\,L}_{\rm sc}$ and $\mathbf{I}^{C,\,L}_{\rm i}$ are the Stokes parameters of scattered and incident radiation referred to the scattering plane containing the directions of propagation of the incident beam and scattered radiation.

Using Eq. (6), it is possible to derive the relations between the scattering matrices of Eq. (8)

$$\mathbf{Z}^{C}(\theta) = \mathbf{A}\,\mathbf{Z}^{L}(\theta)\,\mathbf{A}^{-1}\ . \tag{9}$$

Let us define

$$\mathbf{F}^{C,\,L}(\theta) = \frac{4\pi}{C_{\rm scat}}\,\mathbf{Z}^{C,\,L}(\theta)\ , \tag{10}$$

where

$$C_{\rm scat} = \int\limits_{4\pi} Z^{L}_{11}(\theta)\,{\rm d}\Omega\ , \tag{11}$$

$F^{L}_{11}(\theta)$ is the scattering phase function which satisfies the normalization condition

$$\frac{1}{4\pi}\int\limits_{4\pi} F^{L}_{11}(\theta)\,{\rm d}\Omega = 1\ . \tag{12}$$

For the given volume scattering element the scattering matrix in the LP–representation has the form[7]

$$\mathbf{F}^{L} = \begin{bmatrix} a_1(\theta) & b_1(q) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}, \tag{13}$$

and in the CP–representation[5,8–10] according to Eq. (9) it has the form

$$\mathbf{F}^{C} = \{F^{C}_{mn}\} = \frac{1}{2}\begin{bmatrix} a_2+a_3 & b_1+ib_2 & b_1-ib_2 & a_2-a_3 \\ b_1+ib_2 & a_1+a_4 & a_1-a_4 & b_1-ib_2 \\ b_1-ib_2 & a_1-a_4 & a_1+a_4 & b_1+ib_2 \\ a_2+a_3 & b_1-ib_2 & b_1+ib_2 & a_2+a_3 \end{bmatrix}, \tag{14}$$

$m, n = 2, 0, -0, -2$.

Following Refs. 5 and 8, the elements of the scattering matrix given by Eq. (14) are expanded into a series in generalized spherical functions[11]

$$F^{C}_{m\,n} = \sum_{s=s\,m}^{\infty} g^{s}_{m\,n}\,P^{s}_{m\,n}(\cos\theta)\ ,\quad m,\ n = 2,\ 0,\ -0,\ -2\ ,$$

$$s\,m = \max(|m|,\ |n|)\ . \tag{15}$$

The coefficients of expansion possess the following properties of symmetry[5]:

$$g^{s}_{m\,n} = g^{s}_{n\,m} = g^{s}_{-m-n}\ ,\ g^{s}_{20} = g^{s\,*}_{2-0}\ , \tag{16}$$

where $g^{s}_{m\,n}$ and g^{s}_{m-n} are real numbers.

The CP–representation has the following advantages over the conventional LP–representation:

1) the existence of expansion (15) for the given volume scattering element, and

2) in the CP–representation the transformation matrix of the Stokes parameters for rotation of the reference plane through the angle α is diagonal[5,10]

$$\mathbf{I}^{C}(\alpha) = \mathbf{L}_{C}(\alpha)\,\mathbf{I}^{C}(0)\ ,$$

$$I_n(\alpha) = {\rm e}^{in\alpha}\,I_n(0)\ ,\quad n = 2,\ 0,\ -0,\ -2\ , \tag{17}$$

where the angle is counted off clockwise from the direction of propagation.

The Stokes parameters of a local incident beam referred to a meridian plane containing the propagation direction can be transformed into the Stokes parameters of scattered radiation referred to a meridian plane containing the scattering direction in the following way (see Fig. 1):

1) transformation of the Stokes vector of a local incident beam from the meridian to scattering plane,

2) finding of the Stokes vector of scattered radiation,

3) transformation of the Stokes vector from the scattering to meridian plane.

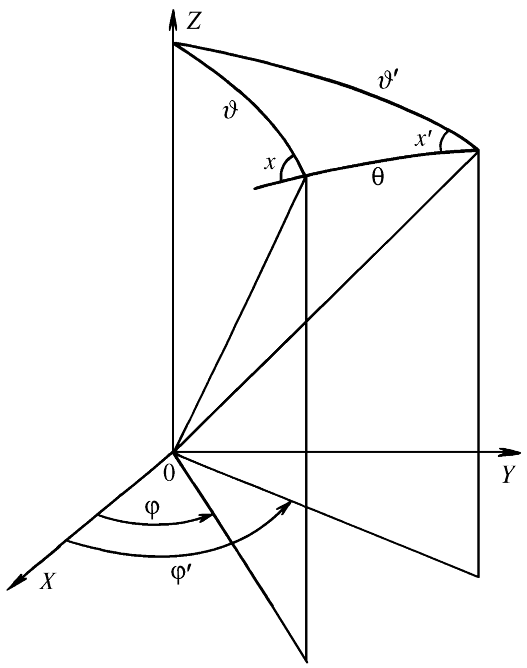

FIG. 1.

On account of Eqs. (8), (10), (15), and (17), the transformation has the form

$$\mathbf{I}^{C}_{\rm sc}(\upsilon,\ \varphi;\ \upsilon',\ \varphi') = r^{-2}\,\mathbf{L}_{C}(-\chi)\,\mathbf{Z}^{C}(\theta)\,\mathbf{L}_{C}(\chi')\,\mathbf{I}^{C}_{\rm i}(\upsilon',\ \varphi')\ ,$$

$$I^{\rm sc}_{m}(\upsilon,\ \varphi;\ \upsilon',\ \varphi') = r^{-2}\,\frac{C_{\rm scat}}{4\pi}\ \times$$

$$\times \sum_{n=2,0,-0,-2}\left\{\sum_{s=sm}^{\infty} g^{s}_{mn}\,{\rm e}^{-im\chi}\,P^{s}_{mn}(\cos\theta)\,{\rm e}^{in\chi'}\right\} I^{\rm i}_{n}(\upsilon',\ \varphi')\ ,$$

$$m = 2,\ 0,\ -0,\ -2\ , \tag{18}$$

where $I_n^{\rm i}$ and $I_m^{\rm sc}$ are the Stokes parameters of a local incident beam and scattered radiation, respectively.

The addition theorem for generalized spherical functions[11]

$$e^{-im\chi}\, P^s_{mn}(\cos\theta)\, e^{in\chi'} =$$

$$= \sum_{q=-s}^{s} (-1)^q\, P^s_{mq}(\cos\upsilon)\, P^s_{qn}(\cos\upsilon')\, e^{iq(\varphi-\varphi')} \quad (19)$$

makes it possible to exclude the variables χ, θ, and χ′ from formula (18).

For focused or divergent incident beam propagating in directions confined to a conic solid angle Ω′, the Stokes parameters of scattered radiation, with allowance for the property of additivity, have the form

$$I^{\rm sc}_m(\upsilon, \varphi; \upsilon'_0) = \frac{1}{\int\limits_{\Omega'} I^{\rm i}(\upsilon', \varphi')\, d\omega'} \int\limits_{\Omega'} I^{\rm sc}_m(\upsilon, \varphi; \upsilon', \varphi')\, d\omega', \quad (20)$$

where $m = 2, 0, -0, -2$ and $I^{\rm i}(\upsilon', \varphi')$ is the local beam intensity.

An infinitely small quantity[12] $d\Phi = I\, d\sigma$ is proportional to the radiant power transported by a pencil of rays in directions confined to an element of solid angle dω, where $d\sigma = r^2 d\omega$ is an element of area the solid angle dω cuts out from the sphere of radius r and I is the radiant intensity.

In the subsequent treatment

$$\Phi = \int\limits_{\Omega} I\, r^2\, d\omega$$

is the radiant flux propagating within the solid angle Ω.

The normalization factor in Eq. (20) is a flux of incident radiation. It should be noted that according to this definition the flux of a parallel beam is zero. In this case the limit $\Omega' \to 0$ must be taken in Eq. (20) finally resulting in formula (8) (with an accuracy of a factor being equal to the incident radiant intensity) and coinciding with it for a unit intensity of the incident beam.

By normalizing Eq. (20) with allowance for the condition of normalization of the scattering phase function given by Eq. (12), after the use of Eqs. (18) and (19) and substitution into Eq. (20), we obtain

$$\hat{I}^{\rm sc}_m(\upsilon, \varphi; \upsilon'_0) = \frac{1}{\int\limits_{\Omega'} I^{\rm i}(\upsilon', \varphi')\, d\omega'} \left\{ \int\limits_0^{2\pi} d\varphi' \int\limits_0^{\upsilon'_0} d\upsilon'_0 \sin\upsilon' \times \right.$$

$$\times \sum_{n=2,0,-0,-2} \left\{ \sum_{s=sm}^{\infty} g^s_{mn} \sum_{q=-s}^{s} (-1)^q \times \right.$$

$$\left. \left. \times P^s_{mq}(\cos\upsilon)\, P^s_{qn}(\cos\upsilon')\, e^{iq(\varphi-\varphi')} \right\} I^{\rm i}_n(\upsilon', \varphi') \right\},$$

$$m = 2, 0, -0, -2 . \quad (21)$$

Formula (21) takes into account the geometry (divergence or convergence) and structure (intensity and polarization) of the incident beam.

Let us consider some particular cases of formula (21).

1) Incident beam homogeneous in intensity ($I^{\rm i}(\upsilon', \varphi') = I^{\rm i}$ = const) and polarization.

(a) Nonpolarized beam. In CP–representation[5,13] $\mathbf{I}^C_i = (0, \frac{I^{\rm i}}{2}, \frac{I^{\rm i}}{2}, 0)^{\rm T}$. After integration of Eq. (21) over φ′, the terms of the series with $q = 0$ remain nonzero, and Eq. (21) is reduced to a simple expression

$$\hat{I}^{\rm sc}_m(\upsilon, \varphi; \upsilon'_0) = \frac{1}{2(1-\cos\upsilon'_0)} \times$$

$$\times \sum_{s=|m|}^{\infty} (g^s_{m0} + g^s_{m-0})\, P^s_{m0}(\cos\upsilon)\, \langle P_s(\cos\upsilon'_0)\rangle , \quad (22)$$

$m = 2, 0, -0, -2$,

$$\langle P_s(\cos\upsilon'_0)\rangle = \int\limits_{\cos\upsilon'_0}^{1} P_s(x)\, dx =$$

$$= \begin{cases} 1 - \cos\upsilon'_0 , & s = 0 , \\ -\sin\upsilon'_0\, P_s^{-1}(\cos\upsilon'_0) , & s > 0 , \end{cases} \quad (23)$$

where P_s and P^m_s are the Legendre polynomials and associated Legendre functions.[11]

The scattering phase function on account of Eqs. (2) and (12) has the form

$$\hat{I}^{\rm sc}(\upsilon, \varphi; \upsilon'_0) = \hat{I}^{\rm sc}_0 + \hat{I}^{\rm sc}_{-0} =$$

$$= \frac{1}{1-\cos\upsilon'_0} \sum_{s=0}^{\infty} a^s_1\, P_s(\cos\upsilon)\, \langle P_s(\cos\upsilon'_0)\rangle , \quad (24)$$

where $a^s_1 = g^s_{00} + g^s_{-00}$ are the coefficients of expansion of the scattering phase function ($F_{11} = a_1$(13)) in the Legendre polynomials.[5,9,10]

(b) For either sense of polarization of the incident beam, the Stokes vector parameters in the CP–representation are equal to $(0, I^{\rm i}, 0, 0)^{\rm T}$ and $(0, 0, I^{\rm i}, 0)^{\rm T}$, respectively, and those corresponding to Eq. (21) are

$$\hat{I}^{\rm sc}_m(\upsilon, \varphi; \upsilon'_0) = \frac{1}{1-\cos\upsilon'_0} \sum_{s=|m|}^{\infty} g_{m0}\, P^s_{m0}(\cos\upsilon)\, \langle P_s(\cos\upsilon'_0)\rangle ,$$

$$\hat{I}^{\rm sc}_m(\upsilon, \varphi; \upsilon'_0) = \frac{1}{1-\cos\upsilon'_0} \sum_{s=|m|}^{\infty} g_{m-0}\, P^s_{m0}(\cos\upsilon)\, \langle P_s(\cos\upsilon'_0)\rangle, \quad (25)$$

$m = 2, 0, -0, -2$.

2. Unpolarized incident beam ($I^{\rm i}(\upsilon', \varphi') = I^{\rm i}(\upsilon')$) inhomogeneous in intensity.

Let us expand the function $I^{\rm i}(\upsilon')$ into a series in the Legendre polynomials

$$I^{i}(\upsilon') = \sum_{s=0}^{\infty} a_s P_s(\cos \upsilon') . \quad (26)$$

On account of the formula[11,14]

$$P^{s}_{mn}(\cos \theta)\, P^{s'}_{m'n'}(\cos \theta) =$$

$$= \sum_{s''=|s-s'|}^{s+s'} C^{s''\, m+m'}_{s\, m\, s'\, m'}\, C^{s''\, n+n'}_{s\, n\, s'\, n'}\, P^{s''}_{m+m'n+n'}(\cos \theta) = , \quad (27)$$

where $C^{JM}_{j\, m\, j_1\, m_1}$ are the Clebsch–Gordan coefficients,[14] we obtain

$$\hat{I}^{sc}_{m}(\upsilon, \varphi; \upsilon'_0) = \left\{ \sum_{s=|m|}^{\infty} \frac{1}{2} (g^{s}_{m0} + g^{s}_{m-0})\, P^{s}_{m0}(\cos \upsilon) \times \right.$$

$$\left. \times \sum_{s'=0}^{\infty} a_{s'} \sum_{s''=|s-s'|}^{s+s'} [C^{s''0}_{s0\, s'0}]^2 \langle P_{s''}(\cos \upsilon'_0)\rangle \right\} \times$$

$$\times \left\{ \sum_{s'=0}^{\infty} a_{s'} \langle P_{s'}(\cos \upsilon'_0)\rangle \right\}^{-1} , \quad (28)$$

$m = 2, 0, -0, -2$.

In formulas (22), (24), (25), and (28) the Stokes parameters of scattered radiation are implicit functions of φ referred to the meridian plane, and their dependence is determined by this meridian plane containing the direction (υ, φ). Table I lists the results of calculation by formula (24) for spherical particles with the index of refraction $m = 1.33$ and diffraction parameter $\rho = 10$ and 50 for different geometry of the incident beam.

In Refs. 3–4, using the generalized spherical functions the analytical expressions were obtained for the radiant flux scattered by a volume scattering element within different solid angles in the case of focused or divergent incident beam.

It should be noted that analogous problem for a single particle requires that an amplitude matrix (Jones matrix) be used,[13] since in this case local beams scattered by individual particles are coherent and the complex amplitudes (Jones matrices) rather than the Stokes parameters (light scattering matrices) should be summed.[13] The exception is the case of a parallel incident beam or an arbitrary polarized plane electromagnetic wave.[3]

The analytical expressions obtained enable one to study the effect of the geometry and structure of the incident beam. Moreover, the knowledge of the expansion coefficients in Eq. (15) (see Refs. 9, 10, 15, and 16) makes the subsequent analysis simpler and minimizes the volume of calculations.

TABLE I.

	υ'_0, deg			
θ°	0°	1°	5°	10°
	ρ = 10			
0	64.7883	64.3951	55.6810	35.8752
10	15.5898	15.6271	16.3738	17.2778
20	4.6803	4.6763	4.6398	5.0595
30	4.0137	4.0184	4.0966	4.0485
40	1.9159	1.9118	1.8364	1.8106
50	8.8641(–1)*	8.9101(–1)	9.8473(–1)	1.1254
60	6.7680(–1)	6.7403(–1)	6.1828(–1)	5.4410(–1)
70	2.3923(–1)	2.4108(–1)	2.7974(–1)	3.4629(–1)
80	2.3121(–1)	2.3068(–1)	2.1972(–1)	2.0321(–1)
90	1,5195(–1)	1.5172(–1)	1.4743(–1)	1.4316(–1)
100	6.8309(–2)	6.9105(–2)	8.5518(–2)	1.1246(–1)
110	1.4989(–1)	1.4909(–1)	1.3314(–1)	1.1055(–1)
120	1.2976(–1)	1.3025(–1)	1.4005(–1)	1.5202(–1)
130	1.4838(–1)	1.4857(–1)	1.5342(–1)	1.6951(–1)
140	3.4732(–1)	3.4634(–1)	3.2479(–1)	2.7858(–1)
150	1.8110(–1)	1.8265(–1)	2.1547(–1)	2.7596(–1)
160	4.4452(–1)	4.4252(–1)	4.0075(–1)	3.2824(–1)
170	2.5417(–1)	2.5613(–1)	2.9592(–1)	3.5381(–1)
180	2.5432(–1)	2.5207(–1)	2.1134(–1)	1.9817(–1)
	ρ = 50			
0	1244.47	1115.16	194.655	63.5497
10	11.4185	10.6874	11.0932	32.9131
20	7.4857	6.9109	5.2298	4.7707
30	2.7400	2.6691	2.4212	2.5053
40	1.3182	1.3262	1.3913	1.3932
50	5.3715(–1)	5.6772(–1)	6.2692(–1)	6.7099(–1)
60	3.1315(–1)	2.9715(–1)	2.7959(–1)	2.9123(–1)
70	7.9111(–2)	9.1894(–2)	1.2395(–1)	1.3326(–1)
80	9.5849(–2)	8.6848(–2)	6.9054(–2)	6.8797(–2)
90	2.4738(–2)	2.6777(–2)	3.1191(–2)	3.3794(–2)
100	1.5444(–2)	1.7164(–2)	1.9866(–2)	2.1066(–2)
110	5.4005(–3)	9.7217(–3)	2.0476(–2)	2.6723(–2)
120	2.9429(–2)	3.8332(–2)	5.0239(–2)	4.4060(–2)
130	3.6375(–2)	4.2932(–2)	7.6169(–2)	8.9188(–2)
140	2.4655(–1)	2.2934(–1)	2.1549(–1)	1.8985(–1)
150	1.2251(–1)	1.2867(–1)	1.6019(–1)	1.7927(–1)
160	7.0563(–2)	8.1232(–2)	1.1139(–1)	1.3792(–1)
170	1.3890(–1)	1.4444(–1)	1.6606(–1)	1.7916(–1)
180	2.0613(–1)	1.8439(–1)	3.3481(–1)	2.0715(–1)

*8.8641(–1) = $8.8641 \cdot 10^{-1}$

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