APPLICATION OF THE CONVEX BOOLEAN PROGRAMMING TECHNIQUES TO THE PROBLEM OF OPTIMAL ARRANGEMENT OF THE CONTROL ACTUATORS IN FLEXIBLE MIRRORS

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An approach, which is based on the convex Boolean programming technique, is proposed for solution of the problem of optimal arrangement of control actuators in flexible mirrors.

Recently adaptive optics have been widely used for improving the quality of reception of optical signals passing through the turbulent atmosphere.[1–3] When designing the systems of adaptive optics (such as the flexible membrane and plate mirrors) one of the important problems is determination of the optimal arrangement of control actuators. The quality of an optical system depends to a large extent on how successfully the actuators are arranged.

The approximate methods of solution of such problems are well known.[4] The drawback of the approximate methods is the difficulty of estimating how close are the derived and optimal solutions. In this paper an approach is proposed that enables one to derive the exact solution for the selected criterion of optimization or the approximate solution with an assured relative or absolute accuracy (according to the criterion of quality).

1. When stating the problem we shall generally follow a description of the problem of the phase distortion correction given in Ref. 4. The wave–front distortion is characterized by the function $\varphi(\mathbf{r})$. On the mirror surface we choose a grid of coordinates of the acceptable points of arrangement of the control actuators (N nodel points). Let us characterize the error in the wave–front correction by the quantity

$$\Delta^2 = \frac{1}{s} \int_{X} \gamma^2(\mathbf{r}) \left[\varphi(\mathbf{r}) - \sum_{i=1}^{N} b_i^{1/2} u_i R_i(\mathbf{r}) \right]^2 \mathrm{d}^2\mathbf{r} \, . \tag{1}$$

In what follows for brevity we will omit the area of integration Ω in the integrals.

In formula (1) Ω is the area of the wave–front correction, u_i is the value of the control action, $R_i(r)$ is the response function, b_i is the indicator variable taking the values of 0 and 1 (the control actuator is positioned in the ith grid node (1) and otherwise (0)), $\gamma^2(r)$ is the weight function (which can be equal to unity in particular case) characterizing the importance of the correction errors in a respective zone of the mirror, and $1/s$ is the normalization factor, where $s = \int \gamma^2(r)\mathrm{d}^2r$.

It is assumed that for the fixed arrangement of the actuators the control action is determined by minimizing the function

$$f(\mathbf{u}) = \Delta^2(\mathbf{u};\, \mathbf{b}) + \lambda\, \mathbf{u}^{\mathrm{T}}\mathbf{u} \, , \tag{2}$$

where $\mathbf{b}$ is the vector of the Boolean variables $\mathbf{b} = (b_1, \ldots, b_N)$, $\mathbf{u}$ is the vector of the control actions $\mathbf{u} = (u_1, \ldots, u_N)$, and T denotes transposition.

Such a statement is idealized, of course, since it presumes the observation of the wave–front phase over the entire aperture as well as the possibility of instantaneous correction of distortions. However for the problem of optimal arrangement of the actuators this statement is quite justified.

The first term in Eq. (2) characterizes the error in correction of the phase distortions and the second term is the addition limiting the great values of the control actions. The value of the limiting addition is controlled by the parameter λ and, depending on anyone's preference, it can be reduced to zero.

The optimal control action $\mathbf{u}^*$ at the fixed b can be found as a solution of the equation derived by setting the derivatives of the function f over all the components u equal to zero.

By omitting the calculations we can write the relation for the optimal control

$$\mathbf{u}^* = (\lambda\, I + B\, D\, B)^{-1}\, B\, \mathbf{c} \, ,$$

where $B = \mathrm{diag}\left(b_1^{1/2}, \ldots, b_N^{1/2}\right)$ and the components of the vector $\mathbf{c}$ and matrix D are determined by the formulas

$$c_i = \int \gamma^2(\mathbf{r})\, \varphi(\mathbf{r})\, R_i(\mathbf{r})\, \mathrm{d}^2\mathbf{r} \, ; \quad D_{ij} = \int \gamma^2(\mathbf{r})\, R_i(\mathbf{r})\, R_j(\mathbf{r})\, \mathrm{d}^2\mathbf{r} \, .$$

We take the criterion of optimization of actuator arrangement in the form

$$J(\mathbf{b}) = \langle \Delta^2(\mathbf{u}^*,\, \mathbf{b}) \rangle \, ,$$

where the angular brackets denote averaging over the ensemble of realizations.

By substituting the relation for $\mathbf{u}^*$ into formula (1) and after several transformations we have

$$J(\mathbf{b}) = \frac{1}{s} \int \gamma^2(\mathbf{r}) \langle \varphi^2(\mathbf{r}) \rangle\, \mathrm{d}^2\mathbf{r} - \langle \mathbf{c}^{\mathrm{T}} B\, V\, B\, \mathbf{c} \rangle - \lambda \langle \mathbf{c}^{\mathrm{T}} B\, V\, V\, B\, \mathbf{c} \rangle \, ,$$

where $V = (\lambda\, I + B\, D\, B)^{-1}$.

Using the calculation operator of the matrix trace tr and introducing the notation $Q = \langle \mathbf{c}\mathbf{c}^{\mathrm{T}} \rangle$ we can write down

$$J(\mathbf{b}) = \frac{1}{s} \int \gamma^2(\mathbf{r}) \langle \varphi^2(\mathbf{r}) \rangle\, \mathrm{d}^2\mathbf{r} - \mathrm{tr}\,(QL) \, ,$$

where $L = B\, V\, (I + \lambda V)\, B$.

The elements of the matrix Q are derived by the formula

$$Q_{ij} = \int \int \gamma^2(\mathbf{r})\, \gamma^2(\rho) \langle \varphi(\mathbf{r})\, \varphi(\rho) \rangle\, R_i(\mathbf{r})\, R_j(\rho)\, \mathrm{d}^2\mathbf{r}\, \mathrm{d}^2\rho \, .$$

The problem of optimal arrangement of actuators is written in the form

$$\min J(\mathbf{b}) \, ; \qquad \mathbf{b} \in \Psi \, ; \qquad \mathbf{b} \in \{0,\, 1\}^N \, . \tag{3}$$

The set Ψ for the simplest case has the form

$$\Psi = \left\{ \mathbf{b} : \sum_{i=1}^{N} b_i = M \right\}$$

that corresponds to the solution of the problem of optimization when there are M actuators. It should be noted that in the general case Ψ can be determined by the system of linear equalities and (or) inequalities corresponding to the conditions of the problem.

2. In this section when studying the properties of J relative to the optimized variable we consider $\mathbf{b}$ as the continuous vector variable $\mathbf{b} \ge 0$.

Let us introduce the notation

$$W = B\,V\,B = (\lambda\,F^{-1} + D)^{-1} = D^{-1} - D^{-1}\left(\frac{1}{\lambda}\,F + D^{-1}\right)^{-1} D^{-1} ,$$

where $F = \mathrm{diag}\,(b_1, \ldots, b_N)$.

After simple transformations for L we derive

$$L = 2\,W - W\,D\,W .$$

Let $\delta\mathbf{b}$ be the small variation of the independent variable. We consider the first variations of $L(\mathbf{b})$:

$$\delta L = 2\,\delta\,W - \delta\,W\,D\,W - W\,D\,\delta\,W ;$$

$$\delta^2 L = \delta^2\,W\,D\,(D^{-1} - W) + (D^{-1} - W)\,D\,\delta^2\,W - 2\,\delta\,W\,D\,\delta\,W .$$

For variations of $W(\mathbf{b})$ we have

$$\delta W = \frac{1}{\lambda} D^{-1}\,Z\,\delta\,F\,Z\,D^{-1} ;$$

$$\delta^2 W = -\frac{2}{\lambda^2} D^{-1}\,Z\,\delta\,F\,Z\,\delta\,F\,Z\,D^{-1} \le 0 ,$$

where $Z = \left(\frac{1}{\lambda}\,F + D^{-1}\right)^{-1}$.

Taking into account that $D^{-1} - W$, $-\delta^2 W$, and D are the positive definite matrices we obtain the inequality $\delta^2 L \le 0$, whence it follows that

$$\delta^2 J = -\,\mathrm{tr}\,(Q\,\delta^2\,L) \ge 0 ,$$

i.e., $J(b)$ is the convex function with respect to $\mathbf{b} \ge 0$.

3. By studying the properties of $J(\mathbf{b})$ we find out that Eq. (3) is the problem of the convex Boolean programming. Effective calculational algorithms can be used for solving such problems. We use an algorithm taken from Ref. 5 which after its simple modification enables one to seek for not only the exact solution, but also approximate solutions with the assured absolute or relative accuracy. The scheme of the algorithm is given below.

Let the initial point $\mathbf{b}^0 \in \Psi$ be available and ε be a prescribed absolute value of the tolerance for the accuracy of a search (with respect to the criterion of quality). Let $m = J(\mathbf{b}^0) - \varepsilon$. The variable being the counter of iterations is assigned an initial value of $i = -1$.

(1) Assume that $i = i + 1$ and draw the tangent hyperplane to $J(\mathbf{b})$ at the point $\mathbf{b}^i$

$$g_i(\mathbf{b}) = J(\mathbf{b}^i) + \sum_{j=1}^{N} \left.\frac{\partial J(\mathbf{b})}{\partial b_j}\right|_{\mathbf{b}=\mathbf{b}^i} (b_j - b_j^i) .$$

(2) Determine the set

$$M_i = \{\mathbf{b} : g_0(\mathbf{b}) < m \,;\, \ldots \,;\, g_i(\mathbf{b}) < m \,;\, \mathbf{b} \in \Psi\} .$$

(3) Find the value k so that

$$J(\mathbf{b}^k) = \min\,[\,J(\mathbf{b}^0), \ldots , J(\mathbf{b}^i)] .$$

Assume $m = J(\mathbf{b}^k) - \varepsilon$.

(4) If the set M_i is empty then the solution of problem (3) with the assured absolute accuracy ε is $\mathbf{b}^k$.

(5) If M_i is not empty, then the solution $\mathbf{b}^{i+1}$ of the problem of the linear Boolean programming can be sought for

$$\min g_i(\mathbf{b}) \,;\ \mathbf{b} \in M_i \,;\ \mathbf{b} \in \{0, 1\}^N .$$

(6) Go again to item 1.

Let us note that in item 5 as $\mathbf{b}^{i+1}$ we can take any acceptable point from the set M_i.

The above–described algorithm converges over the finite number of iterations.

When the problem of optimization with the prescribed relative tolerance δ is needed to be solved, in the above–described algorithm it is necessary to set $m = (1 - \delta^*)J(\mathbf{b}^0)$ before entering the cycle and $m = (1 - \delta^*)J(\mathbf{b}^k)$ in the body of the iterating cycle, where $\delta^* = \delta/(1 + \delta)$.

4. Let us consider the problem of optimal arrangement of the M control actuators on the flexible mirror when

$$\gamma^2(\mathbf{r}) = \exp\{-|\mathbf{r}|^2/r_c^2\} ;$$

$$R_i(\mathbf{r}) = \exp\{-|\mathbf{r} - \mathbf{r}_i|^2/r_R^2\} ;$$

$$\langle\varphi(\mathbf{r})\,\varphi(\boldsymbol{\rho})\rangle = \sigma_\varphi^2 \exp\{-|\mathbf{r} - \boldsymbol{\rho}|^2/r_0^2\} ,$$

where $\mathbf{r}_i (i = 1, \ldots, N)$ are the coordinates of the nodel points, r_0 is the radius of the Fried correlation for fluctuations of the wave front.

In order to have the possibility of performing the analytical calculations of elements of the matrices Q and D we assume that the circled area Ω has the radius much greater than r_γ. The elements of the matrices Q and D can be derived by the formulas

$$Q_{ij} = \frac{\pi^2\sigma_\varphi^2}{\eta} \exp\left\{-\frac{1}{\eta\,r_R^4}(r_i^2 + r_j^2)(\eta\,r_R^2 - \alpha) - \frac{2}{r_0^2} r_i\,r_j \cos(\theta - \theta_j)\right\};$$

$$D_{ij} = \frac{\pi\,(r_R^2 + r_\gamma^2)}{r_R^2 + 2\,r_\gamma^2} \exp\left\{-\frac{1}{r_R^2\,(r_R^2 + 2\,r_c^2)} \times \right.$$

$$\left. \times \left[(r_i^2 + r_j^2)\,(r_R^2 + r_\gamma^2) - 2\,r_i\,r_j\,r_\gamma^2 \cos(\theta_i - \theta_j)\right]\right\} ,$$

where $\alpha = \dfrac{1}{r_0^2} + \dfrac{1}{r_R^2} + \dfrac{1}{r_\gamma^2}$; $\eta = \alpha^2 - \dfrac{1}{r_0^4}$.

The grid of acceptable nodel points ($N = 31$) is taken as in Ref. 4 on the "cobweb" formed by three equidistant concentric circles and twelve rays issued out of the center at equal angles. We take the central point, the odd points on intersection of the first (inner) circle with the rays, and all the points of intersection of the second and third circles with the rays as the nodel points. The radii of the circles are equal to 2, 4, and 6, respectively (in the given case the dimensionless values are considered).

In calculations the following numerical values of the parameters are taken: $\sigma_{\varphi}^2 = 0.01$, $r_0 = 15$, $r_\gamma = 6$, $r_R = 10$, and $\lambda = 0.3$.

The problem of arrangement of four actuators is solved.

The proposed approach to solution of the problem of optimal arrangement of actuators was studied by a computer of IBM PC AT series. In the algorithm of the convex Boolean programming the number of iterations is turned out to be dependent strongly on the value of the tolerance for the solution accuracy (the "rougher" is the solution, the less is the number of iterations).

Table I represents the results obtained for different values of the relative tolerance at the initial point b^0 corresponding to the nodel points numbered 2, 3, 5, and 6. The value of the criterion of quality J^* corresponds to the value of the argument b obtained in the search.

TABLE I.

δ(%)	$J^*\cdot 10^3$	Nodel point numbers				Iteration number
0	1.799	21	24	27	30	65
1	1.799	21	24	27	30	48
2	1.799	21	24	27	30	40
5	1.835	22	24	27	30	22
10	1.886	22	23	27	31	8
20	1.928	4	20	24	29	5

It can be seen from Table I that for relative tolerances 1% and 2% the obtained solutions are identical to the exact ones. For a relative accuracy of 10% the solution can be obtained by eight iterations (which is eight times less than the number of iterations in the search for the exact solution), i.e., approximately in an order of magnitude faster than the exact solution.

The process of the search when δ = 1% is shown in Table II.

TABLE II.

Serial number of iterations	$J\cdot 10^3$	Nodel point numbers			
1	2.365	2	3	5	6
2	2.106	23	24	25	30
3	2.496	19	20	28	29
4	2.052	4	18	20	24
5	2.064	11	22	27	28
6	1.976	11	15	30	31
7	1.886	22	23	27	31
8	1.902	13	21	26	29

The search ends after completion of all eight iterations. The point obtained at the seventh iteration is taken as an approximate solution with the assured 10% accuracy.

Let us note that the above–described approach can be used together with other approximate and heuristic algorithms (which have been already known or will be developed in future), in particular, for estimating their accuracy.

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