ON THE TRANSMITTANCE OF A LAYER OF NONSPHERICAL SPATIALLY ORIENTED SCATTERING PARTICLES

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An approximate analytical method for calculating spatial irradiance and relative fluxes in a medium containing nonspherical, spatially oriented particles is developed on the basis of the radiation transfer equation. The accuracy of the given method is evaluated for the simplest model of the optical parameters of a scattering medium with nonspherical spatially oriented particles.

To solve various applied problems in atmospheric optics and climatology one has to know energy parameters of light scattering particles in the visible spectrum. In certain cases the scattering particles are nonspherical and have a dominating spatial orientation (e.g., raindrops, ice crystals, etc.).[1,2] Starting from the equation of radiation transfer, we have developed a new technique for an approximate calculation of spatial irradiance and radiation fluxes in a semi-infinite scattering medium and in a plane-parallel layer containing such particles. A similar problem was treated elsewhere.[3] However, the solution obtained there is poorly suited to practical computations since it requires knowing the eigenfunctions of an integral equation with a kernel, for which the scattering phase function of the nonspherical particles is taken. The computations in this paper are based on the easily calculated scattering parameters.

It is known that only particles of sizes considerably larger than the wavelength of visible light can feature dominating orientations under atmospheric conditions.[2] This fact simplifies our problem because for such large particles their scattering phase function has a sharp maximum in the direction of the incident light wave. In this case the transport approximation can be used to solve an equation of radiation transfer, which provides high accuracy for the spatial irradiance computations.[4,5] Then the scattering phase function has the form

$$\chi(\vec{\Omega}, \vec{\Omega}') = 1 - a(\vec{\Omega}) + 4\pi a(\vec{\Omega})\delta(\vec{\Omega}\cdot\vec{\Omega}') \tag{1}$$

where $\delta(\vec{\Omega}\cdot\vec{\Omega}')$ is the δ-function, and $a(\vec{\Omega})$ is the average cosine of the scattering angle when the incident wave propagates in the direction of the unit vector $\vec{\Omega}$. Let a light wave, monodirected along the z-axis and producing unit irradiance at the upper boundary of the medium, be incident upon a semi-infinite medium bounded by the plane $z = 0$, or upon a layer, limited by the planes $z = 0$ and $z = H$. With (1) taken into account the solution of the transfer equation can then be written as

$$\mu \frac{dI}{dz} + \alpha(\vec{\Omega}) I = \beta(\vec{\Omega})\rho + Q. \tag{2}$$

Here

$\alpha(\vec{\Omega}) = \varepsilon(\vec{\Omega})(1 - \Lambda(\vec{\Omega})a(\vec{\Omega}))$,

$\beta(\vec{\Omega}) = \sigma(\vec{\Omega})\cdot(1 - a(\vec{\Omega}))/4\pi$;

I is the radiance; $\rho = \int\limits_{4\pi} I(\vec{\Omega})d\vec{\Omega}$ is the spatial irradiance; Q is the radiation source function; $\varepsilon(\vec{\Omega})$ is the extinction factor for an elementary volume of the medium; $\sigma(\vec{\Omega})$ is the scattering factor for the same volume; $\Lambda(\vec{\Omega})$ is the photon survival probability, $\Lambda = \sigma/\varepsilon$; $\vec{\Omega} = \{\Theta, \varphi\}$, where Θ is the polar, and φ is the azimuth angles; and $\mu = \cos\Theta$. With the problem statement taken into account the source function Q has the form

$$Q(z, \vec{\Omega}) = [\Theta(z) - \Theta(z - H)]\, e^{-\varepsilon z}\sigma\chi(\vec{\Omega})/4\pi$$

The functions ε, τ, and a, without their arguments, imply that their argument is actually $\vec{\Omega} = \vec{\Omega}_z$, where $\vec{\Omega}_z$ is the unit vector along the z axis.

When we employ the transport approximation, the values of directly transmitted light and of the source function have to be redefined, which results in the expression

$$I_{\text{trans}} = e^{-\varepsilon_T z}\,\delta(\vec{\Omega}\cdot\vec{\Omega}_z),$$

$$Q = \frac{\sigma(1 - a)}{4\pi}\, e^{-\varepsilon_T z}\, [\Theta(z) - \Theta(z - H)], \tag{3}$$

where $\varepsilon_t = \varepsilon(1 - \Lambda a)$. One can transfer from (2) to the respective integral equations for radiance and spatial irradiance[3,8]

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$$I(\vec{\Omega}, z) = \begin{cases} \int\limits_{-\infty}^{z} \frac{\beta(\vec{\Omega})\rho + Q}{\mu} e^{-\frac{\alpha(\Omega)}{\mu}(z-z')} dz', & 0 < \mu \le 1 \\ -\int\limits_{-\infty}^{z} \frac{\beta(\vec{\Omega})\rho + Q}{\mu} e^{-\frac{\alpha(\vec{\Omega})}{\mu}(z-z')} dz', & -1 \le \mu < 0 \end{cases} \tag{4}$$

$$\rho(z) = \int\limits_{-\infty}^{z}\int\limits_{0}^{2\pi}\int\limits_{0}^{1} (\beta(\vec{\Omega})\rho + Q)\frac{1}{\mu} e^{-\frac{\alpha(\vec{\Omega})}{\mu}(z-z')} d\mu d\varphi dz' + \int\limits_{z}^{\infty}\int\limits_{1}^{2\pi}\int\limits_{-1}^{0} (\beta(\vec{\Omega})\rho + Q)\frac{-1}{\mu} e^{-\frac{\alpha(\vec{\Omega})}{\mu}(z-z')} d\mu d\varphi dz' \tag{5}$$

In the visible spectrum $1 - \Lambda \ll 1$ usually. Then $\rho(z')$ slowly changes along z compared with the kernel in (4); the latter displays a sharp maximum at $z = z'$ and bolls off quickly at higher $|z - z'|$. Based on the foregoing comments the variable $\rho(z')$ can be expanded in Taylor's series about the point $z = z'$, and, limiting the expression to quadratic terms, we then obtain

$$K_2 = \frac{d^2\rho}{dz^2} - K_1\frac{d\rho}{dz} - \rho(z)(1 - K_0) = P, \tag{6}$$

where

$$K_0 = \frac{1}{4\pi}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1} \frac{\sigma(\vec{\Omega})(1-a(\vec{\Omega}))}{\alpha(\vec{\Omega})} d\mu d\varphi,$$

$$K_1 = \frac{1}{4\pi}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1} \frac{\sigma(\vec{\Omega})(1-a(\vec{\Omega}))\mu}{\alpha^2(\vec{\Omega})} d\mu d\varphi,$$

$$K_2 = \frac{1}{4\pi}\int\limits_{0}^{2\pi}\int\limits_{-1}^{1} \frac{\sigma(\vec{\Omega})(1-a(\vec{\Omega}))\mu^2}{\alpha^3(\vec{\Omega})} d\mu d\varphi,$$

$$P = -\int\limits_{0}^{2\pi}\left[\int\limits_{-\infty}^{z}\int\limits_{0}^{1} \frac{1}{\mu}\cdot Q(z', \mu, \varphi)\cdot e^{\frac{-\alpha(\vec{\Omega})}{\mu}(z-z')} d\mu dz' - \int\limits_{z}^{\infty}\int\limits_{-1}^{0} \frac{1}{\mu}\cdot Q(z', \mu, \varphi)\cdot e^{\frac{-\alpha(\vec{\Omega})}{\mu}(z-z')} d\mu dz'\right] d\varphi.$$

The expression (6) yields the extinction parameters in the deep layers for radiation propagating through the medium

$$\gamma_{1,2} = \left(K_1 \pm \sqrt{K_1^2 + 4K_2(1-K_0)}\right)/2K_2. \tag{7}$$

The plus sign in (7) corresponds to the upward propagating radiation, and the minus sign — to downward.

Recalling that as $\Lambda \to 1$ the value $1 - K_0$ approaches to zero, while K_1 and K_2 approach some nonzero constants in the general case, we conclude that γ_1 does not approach zero, as $\Lambda \to 1$. At the same time it is apparent that the asymptotics $\gamma \sim 1 - \Lambda$, typical for spherical particles, is not valid for γ_1.

All the common models of nonspherical particles (spheroids, cylinders, columnar crystals) have a center of symmetry. Hence, the parameters ε, σ, and a for an elementary scattering volume containing such particles, should possess the following property: $\varepsilon(\bar{\Omega}) = -\varepsilon(\bar{\Omega})$; similar expressions are true for σ and a. Therefore we may further assume that $K_1 = 0$, $\gamma_1 = -\gamma_2 = \gamma = (1 - K_0 / K_2)^{1/2}$.

As the boundary conditions for (6) we take the condition that the flux goes to 0 at the boundary, which is common for the diffusion approximation

$$\int\limits_{0}^{2\pi}\int\limits_{0}^{0} I(\mu, \varphi, z = 0)\, \mu d\mu d\varphi = 0,$$
$$\int\limits_{0}^{2\pi}\int\limits_{0}^{0} I(\mu, \varphi, z = H)\, \mu d\mu d\varphi = 0, \tag{8}$$

In case of a layer the respective condition for $I(\mu, \varphi, z)$ may be found from (3).

The solution of (6) with the source function (3) and boundary conditions (8) is obvious. We omit it here because of its cumbersome form. The corresponding computer calculations do not present any difficulties. The layer reflectance and transmittance are found from the relationships

$$S = \int\limits_{0}^{2\pi}\int\limits_{-1}^{0}\int\limits_{0}^{H} (\beta\rho + Q)\, e^{\frac{\alpha}{\mu}z} dz d\mu d\varphi, \tag{9}$$

$$T = \int\limits_{0}^{2\pi}\int\limits_{0}^{1}\int\limits_{0}^{H} (\beta\rho + Q)\, e^{\frac{\alpha}{\mu}z} dz d\mu d\varphi. \tag{10}$$

We selected as our concrete model to compute the layer transmittance using the above technique and to generally assess the effect of particle orientation upon the radiation transfer the simplest set of parameters of an elementary volume, typical for crystal clouds. In particular, the following model of ice crystal particle orientation in the atmosphere has found wide application to problems of atmospheric optics: the particles are oriented with their principal axes in the horizontal, with their horizontal orientation in terms of angle if being completely purely chaotic.[9–11] Consequently, the parameters ε, σ, a of this model do not depend on the azimuth angle φ of the incident radiation, and may only depend on μ.

As demonstrated by calculations, the parameters a, Λ, and the extinction efficiency factor for a separate particle depend weakly on $\vec{\Omega}$ for ice crystals in the shape of elongated cylinders, provided the cylinder radius is much larger than the wavelength.[6] Independence of the extinction efficiency factor on $\vec{\Omega}$ means that extinction by a particle is proportional to its geometrical cross section in the plane normal to the direction of radiation incidence. Then the elementary volume containing such particles, their axes chaotically oriented in φ, must possess a parameter ε(μ), proportional to the cross-sectional area of such a particle, which is obtained from the initial one by rotating it around its vertical axis. A similar form of the ε(μ) dependence is obtained for an elementary volume containing particles in the shape of oblate spheroids with their symmetry axes oriented vertically.

This reasoning results in the following model of nonsphericity: parameters Λ and *a* do not depend on the radiation incidence direction $\vec{\Omega}$, while the parameter ε is proportional to the cross-sectional area of a spheroid, oblate to the necessary degree.

We introduce the ratio $\Delta = L/M$, where L and M are, respectively, the vertical and horizontal axes of a spheroid possessing a vertical symmetry axis. Let $\varepsilon(\mu = 1) \equiv 1$ for clarity.

The results from a comparison of T, obtained for $H = 2$ and $\Lambda = 0.995$, with their exact values are presented in Table I, where $d = (T_{\text{exact}} - T_{\text{appr}})/T_{\text{exact}} \times 100\%$. The exact values were computed employing the Monte-Carlo technique at a relative rms error of 0.2–0.3%. The scattering phase function model assumed for the Monte-Carlo calculations was the Henji-Greenstein, with the average cosine of the scattering angle $\bar{\mu}$.

An increase of d for lower $\bar{\mu}$ is explained by the decreasing accuracy of the transport approximation in that domain.

As follows from the structure of the solution (5), for large H the suggested technique will have high accuracy when the deep regime prevails for the condition $\gamma \ll \varepsilon_T$. Since it is true for the given model, that

$$\gamma = \sqrt{(1 - 0)(1 - a) \Big/ \frac{1}{2}\int_0^1 \frac{\mu^2}{\varepsilon^2(\mu)}\, d\mu},$$

we have

$$\sqrt{1 - \Lambda} \ll \sqrt{(1 - a)\int_0^1 \frac{\mu^2}{\varepsilon^2(\mu)}\, d\mu}. \tag{11}$$

In the case of a sphere $\Delta = 1$ we obtain $\sqrt{1-\Lambda} \ll \sqrt{(1-a)/3}$. For lower Δ at identical Λ and a the right part of the inequality (11) will grow, i.e., the error would diminish for a larger degree of "oblateness".

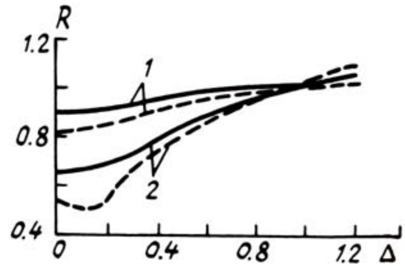

FIG. 1. The value of R(Δ) for Λ = 0.995. Solid lines – μ = 0.9, dashed lines – μ = 0.8, H = 2(1); H = 10(2).

Naturally, certain dependences of Λ and *a* on μ do exist for actual oriented particles. However, considering the weak absorption of light in the visible spectrum, the error of the solution of (5) will differ only slightly from that shown in Table I. To assess applicability of this technique to "thick" layers one has to use the relationship $\gamma \ll \varepsilon_T$, irrespective of the chosen model for Λ, ε and *a*.

TABLE I.

Δ	$\bar{\mu} = 0.9$			$\bar{\mu} = 0.8$		
	$T_{\text{appr.}}$	T_{exact}	d, %	T_{appr}	T_{exact}	d,%
0.1	0.895	0.964	7.7	0.811	0.932	13.0
0.3	0.889	0.953	6.7	0.797	0.902	11.6
0.5	0.886	0.947	6.4	0.789	0.890	11.3
0.8	0.882	0.945	6.7	0.782	0.876	10.7
1.25	0.878	0.936	6.2	0.774	0.863	10.5

Let us consider the behavior of T within this simplest model of particle-nonsphericity. We varied the parameters Λ and *a* within the limits Λ = 0.99–1.0, *a* = 0.8–0.95, which are typical for large atmospheric particles scattering in the visible spectrum. From these calculations one can conclude that the following is valid for $R(\Delta) = T(\Delta)/\bar{T}$, where $\bar{T}$ is the layer transmittance for the chaotic orientation of particles, which in this model is equivalent to the case of considering spherical particles having $\bar{\varepsilon} = \frac{1}{\varepsilon}$, where $\bar{\varepsilon} = 0.5\int_{-1} \varepsilon(\mu)d\mu$: value of R increases with an increase in Δ (see Fig. 1) and with an increase in a (i.e., with an increase in overall particle size). However, for small Δ, when $T \simeq 1$ and is not practically independent of Δ, the value of R is close to the value of $e^{-\sigma(1-a)H}$ equal to the amount of "directly transmitted" light in the transport approximation. The dependence of R on Λ is observed only for large layer depths H.

REFERENCES

1. A. Ishimaru, *Wave Propagation and Scattering in Randomly 1nhomogeneous Media,* Vol. 1, Academic Press, New York, 1978.
2. C.F. Bohren and D.R. Huffman, *Absorption and Scattering of Light by Small Particles*, Wiley. New York. 1983.
3. K. Furutsu, J. Math. Phys. **21**, No. 4, 765 (1980).
4. O.V. Manuilova and L. M. Romanova. Izv. Akad. Nauk SSSR, Ser. FAO **23**, No. 3. 269 (1987).
5. O.V. Bushmakova, L.P. Bass and E.P. Zege, Izv. Akad Nauk SSSR, Ser. FAO **8**, No. 10, 114 (1972).
6. A.G. Petrushin, Izv. Akad. Nauk SSSR, Ser. FAO **22**, No. 12, 1293 (1986).
7. E.P. Zege, A.P. Ivanov and I.L. Katsev, *Image Transfer in Scattering Media* (Nauka i Tekhnika, Minsk, 1985).
8. B. Davison, *Neutron Transport Theory*, Clarendon Press, Oxford, 1957.
9. V.N. Lopatin and L.E. Paramonov, *Studies of Angular Dependence of NSR Elements by Spheroid Particles Systems*, Preprint (Institute of Physics, Siberian Branch of the Academy of Sciences of the USSR, Krasnoyarsk. 1987).
10. K–N. Liou, *Foundations of the Atmospheric Radiation Transfer Processes* (Gidrometeoizdat, Leningrad, 1984).
11. S. Asano, Appl. Opt. **22**, No. 9, 1390 (1983).