Source reconstruction from its noisy and incomplete image

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A convergent iterative algorithm is proposed for finding a regularized solution for the problem of source reconstruction from its noisy and incomplete image. This problem can be reduced to the problem of finding a common point of convex sets and solved by the projection method proposed in my earlier papers.

1. Formulation of the problem

The intensity distribution $y(t_1, t_2)$ in the image drawn by an isoplanatic optical system is related to the intensity of an extended source of incoherent light $x(t_1, t_2)$ through the convolution integral[1]:

$$\int\limits_{-\infty}^{\infty}\int h(t_1-\tau_1, t_2-\tau_2)\,x(\tau_1,\tau_2)\,\mathrm{d}\tau_1\mathrm{d}\tau_2 = y(t_1,t_2)\,,\quad (1)$$

$$-\infty < t_1, t_2 < \infty\,,$$

where $h(t_1, t_2) \ge 0$ is the point spread function. A short-cut form of the equation (1) can be written as $h{*}x = y$ or $Ax = y$. The functions h and x are believed to be summable functions in the plane of the variables t_1 and t_2 ($h, x \in L_1$), and the function y is thought to be a square summable function ($y \in L_2$).

Assume that the right-hand side of Eq. (1) is known on a limited set ω, which determines the region of image observation (measurement). The observation accuracy is specified by the condition $y \in Y$, where Y is a closed convex set. An example of such a set is

$$Y = \left\{ y \in L_2 : y = \tilde{y} - u,\ \|u\|_\omega^2 = \iint\limits_\omega u^2(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 \le \delta^2 \right\},\qquad (2)$$

where $\tilde{y} \ge 0$ is the observed noisy image; u is the unknown additive noise.

The function $x(t_1, t_2) \ge 0$ is assumed finite with the carrier from the set ω_0.

Consider Eq. (1) as an equation for x and y. A solution to Eq. (1) is any pair of the functions $(x, y) \in H = L_2 \times L_2$ satisfying this equation.

Define the scalar product on H

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle =$$

$$= \int\limits_{-\infty}^{\infty}\int x_1(t_1,t_2)\,x_2(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 +$$

$$+ \int\limits_{-\infty}^{\infty}\int x_1(t_1,t_2)\,x_2(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2,$$

and thus transform H into the Hilbert space.

The set of solutions of Eq. (1) is a closed linear manifold in H:

$$V = \{(x, y) \in H : Ax = y\}.\qquad (3)$$

The accepted restrictions on the solution determine the closed convex set

$$V_1 = \{(x, y) \in H : x(t_1, t_2) \ge 0,\ (t_1, t_2) \in \omega_0;$$

$$x(t_1, t_2) \equiv 0,\ (t_1, t_2) \notin \omega_0;\ y \in Y\}.\qquad (4)$$

The problem is to find a solution to Eq. (1) satisfying the restriction V_1. This problem can be treated as finding a common point of the closed convex sets (CPCCS): $(x, y) \in VV_1$. In the general case, restrictions can be different and more numerous. Assume that it was found that the *a priori* solution satisfies m restrictions specified by closed convex sets $V_1, V_2, \ldots, V_m \subset H$. Then any point (x, y) of the set VV_0, $V_0 = \bigcap\limits_{s=1}^{m} V_s$, if it is not empty, is a solution of Eq. (1).

Among the elements of the set VV_0, let us separate the one (x^*, y^*), which has the minimum norm

$$\|(x^*, y^*)\|^2 = \langle x^*, x^* \rangle + \langle y^*, y^* \rangle = \min_{(x,y)\in VV_0} \|(x, y)\|^2\,.$$

A pair of functions (x^*, y^*) will be referred to as the normal solution of Eq. (1). It is obvious that $(x^*, y^*) = P_{VV_0}(0, 0)$, where $P_{VV_0}(x, y)$ is the projection operator on the set VV_0 determined by the condition

$$(x^*, y^*) = P_{VV_0}(x, y) : \|(x, y) - (x^*, y^*)\| =$$

$$= \min_{(x', y')\in VV_0} \|(x, y) - (x', y')\|.$$

Our aim is to find an approximate solution $(x^*(\alpha_0), y^*(\alpha_0))$ of Eq. (1) depending on the parameter $\alpha_0 > 0$. This solution continuously depends on the measured image $\tilde{y}$ and its norm in H at $\alpha_0 \to 0$ tends to the normal solution $\|(x^*(\alpha_0), y^*(\alpha_0)) - (x^*, y^*)\| \to 0$ at $\alpha_0 \to 0$. This approximate solution

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will be referred to as the regularized solution of Eq. (1). It must not belong to the set VV_0, though it is in the close proximity of (x^*, y^*) and, consequently, to the set VV_0.

2. Problems in solving the Eq. (1)

Equation (1) is the integral equation of the first-kind of the convolution type, and the difficulties of its solution are well known.[2] Let this equation be solved by the convolution inversion method:

$$x = F^{-1}\,[F(y)/F(h)], \tag{5}$$

where F and F^{-1} are the direct and inverse Fourier transform operators. With the increase of the spatial frequencies $\nu_1^2 + \nu_2^2 \to \infty$, the functions $F(h)$ and $F(y)$ tend to zero and this tendency should be matched. Small distortions of $F(y)$ at large spatial frequencies due to the noise in the image measurements can drastically change the ratio $F(y)/F(h)$ and, consequently, x. To provide for a continuous dependence of the solution on the initial data, the approximate (regularized) solution[2,3]

$$x = F^{-1}\,\{F(y)\,F^*(h)/[\,|F(h)|^2 + \alpha_0\,M(\nu_1, \nu_2)]\} \tag{6}$$

is considered in spite of Eq. (5). In Eq. (6), asterisk denotes complex conjugation, and M is an even positive definite function. To use Eqs. (5) and (6), one should know the image y on the whole plane. Actually, we have an incomplete image measured in the region ω, and the information on the image can also absent at some points of ω. Therefore, the procedure of image pre-processing in ω is used, as well as smooth extrapolation of the image beyond ω (Ref. 4).

We consider incompleteness of the data and the character of noise as restrictions imposed on the solution. The problem can be reduced to finding a solution satisfying these restrictions. Such an approach has become classical in a certain sense.[2,5,6]

The problem of finding the solution to Eq. (1) with restrictions can be treated as a problem of finding the CPCCS, and the latter can be efficiently solved by iteration methods.[5,7] An iteration method called the method of increased dimension (MID) was proposed in Ref. 7. This method is used for the development of a convergent iteration algorithm for finding the regularized solution ($x^*(\alpha_0)$, $y^*(\alpha_0)$), which was considered above.

3. Method of increased dimension

Let us briefly consider the method of increased dimension in accordance with Ref. 7 and prove a theorem needed for obtaining the regularized solution of Eq. (1). Let H be some Hilbert space, x, x_1, …, x_m are points in it, and the closed convex manifold V and the closed convex sets V_1, …, V_m are defined in H. Consider the functional

$$J(x, x_1, \ldots, x_m) = \sum_{s=1}^{m} \alpha_s \|x - x_s\|^2, \quad \alpha_s > 0,$$

$$\alpha_1 + \ldots + \alpha_m = 1,$$

where $x \in V$ and $x_s \in V_s$, $s = \overline{1, m}$. The functional has the minimum at $x = x_1 = \ldots = x_m$. The minimizing sequence (x_n, x_{1n}, …, x_{mn}) specifies approaching points in the sets V, V_1, …, V_m. As these points coincide, the functional has the minimum, therefore it is called the approach functional. Points coincide at the intersection of $VV_0 = VV_1, \ldots, V_m$. This can be used to find CPCCS. The minimizing sequence is constructed by the method of coordinate descend along the variable x and the set of variables x_1, …, x_m.

Let (x_n, x_{1n}, …, x_{mn}) be the nth approximation; and the (n + 1)th one will be constructed following the scheme

$$x_{sn+1} = P_{V_s}\,x_n = P_s\,x_n,\ s = \overline{1, m},$$

$$x_{n+1} = PT\,x_n,\ P = P_V,\ T = I + \lambda\,(\overline{P} - I),$$

$$\overline{P} = \sum_{s=1}^{m} \alpha_s P_s, \quad 0 < \lambda < 2,$$

where I is the identity operator. The operators P and T are nonextending, and P is linear. The set of immovable points of the operator product PT coincides with the set VV_0, and the sequences $\{x_n\}$ and $\{x_{sn}\}$ weakly converge to a point of the set VV_0.

Consider the regularized functional

$$J_1(x, x_1, \ldots, x_m) = \alpha_0 \|x\|^2 + J(x, x_1, \ldots, x_m), \quad \alpha_0 > 0.$$

In the method of coordinate descend for the functional J_1, the approximation x_{sn+1} is calculated as before, and

$$x_{n+1} = P\tilde{T}\,x_n,\ \tilde{T} = I + \lambda\,(\tilde{P} - I),\ \tilde{P} = \overline{P}/(1 + \alpha_0).$$

The operator $P\tilde{T}$ is contracting with the contraction coefficient $[1 - \lambda\alpha_0/(1 + \alpha_0)]$. Therefore, the sequence $\{x_n\}$ starting from any point of the set V converges with the geometric rate to the sole point $x^*(\alpha_0) = P\tilde{T}x^*(\alpha_0) \in V$. Because of the continuity of the projection operators, the norm of the sequence $\{x_{sn}\}$ converges to the point $x_s^*(\alpha_0) = P_s\,x^*(\alpha_0)$.

Theorem. At $\alpha_0 \to 0$ the norm of the immovable point $x^*(\alpha_0)$ of the operator $P\tilde{T}$ tends to the point $x^* = P_{VV_0}0$, i.e., the point with the minimum norm in the set VV_0, if this set is not empty.

Proof. Consider a set of positive numbers $\alpha_{01} > \alpha_{02} > \ldots > \alpha_{0k} > \ldots$, $\alpha_{0k} \to 0$ at $k \to \infty$. Denote

$x_k^* = x^*(\alpha_{0k})$. Then, taking into account the linearity of the operator P and the condition $Px_k^* = x_k^*$, we can find

$$x_k^* = P\tilde{T}\, x_k^* = P\,[x_k^* + \lambda\,(\tilde{P}x_k^* - x_k^*)] =$$

$$= Px_k^* + \lambda\,(P\tilde{P}x_k^* - Px_k^*) = x_k^* + \lambda\,(P\tilde{P}x_k^* - x_k^*),$$

wherefrom

$$P\tilde{P}x_k^* - x_k^* = 0 \text{ and } P\overline{P}x_k^* = (1 + \alpha_{0k})\, x_k^*. \quad (7)$$

The point $(x_k^*,\ P_1\, x_k^*,\ ...,\ P_m\, x_k^*)$ transforms the functional J_1 into the minimum at the direct product $V \times V_1 \times ... \times V_m$, therefore the following inequality is valid:

$$\alpha_{0k}\| x_k^*\|^2 + \sum_{s=1}^{m}\alpha_s\| x_k^* - P_s x_k^*\|^2 \le \alpha_{0k}\| x^*\|^2 +$$

$$+\sum_{s=1}^{m}\alpha_s\| x^* - P_s x^*\|^2 = \alpha_{0k}\| x^*\|^2,$$

wherefrom we can conclude that immovable points x_k^* are restricted to the set

$$\| x_k^*\| \le \| x^*\|.$$

Let $l > k$ and $h = x_l^* - x_k^*$. From Eq. (7) and the condition of nonextending operator $P\overline{P}$, we have the inequality

$$\|(1+\alpha_{0l})x_l^* - (1+\alpha_{0k})\, x_k^*\| =$$

$$= \| P\overline{P}x_l^* - P\overline{P}x_k^*\| \le \| x_l^* - x_k^*\| = \| h\|.$$

The squared left-hand side of this inequality is

$$\|(1+\alpha_{0l})x_l^* - (1+\alpha_{0k})x_k^*\|^2 =$$

$$= \|(\alpha_{0l} - \alpha_{0k})x_k^* + (1+\alpha_{0l})h\|^2 =$$

$$= (\alpha_{0l} - \alpha_{0k})^2\| x_k^*\|^2 + 2(\alpha_{0l} - \alpha_{0k})\times$$

$$\times(1+\alpha_{0l})\mathrm{Re}\langle x_k^*, h\rangle + (1+\alpha_{0l})^2\| h\|^2,$$

therefore it can be presented in the form

$$(\alpha_{0l} - \alpha_{0k})^2\| x_k^*\|^2 + \alpha_{0l}(2 + \alpha_{0l})\| h\|^2 \le$$

$$\le 2(\alpha_{0k} - \alpha_{0l})\mathrm{Re}\langle x_k^*, h\rangle.$$

Thus, we can conclude that $\mathrm{Re}\,\langle x_k^*, h\rangle \ge 0$. But from the equality $\| x_l^*\|^2 = \| x_k^*\|^2 + \| h\|^2 + 2\mathrm{Re}\,\langle x_k^*, h\rangle$ it follows that $\| x_l^*\|^2 \ge \| x_k^*\|^2 + \| h\|^2$. Then we can conclude that the sequence of norms $\{\| x_k^*\|\}$ is nondecreasing, bounded, and has a limit, and the norm

$$\| x_l^* - x_k^*\|^2 = \| h\|^2 \le \| x_l^*\|^2 - \| x_k^*\|^2 \to 0 \text{ at } k \to \infty.$$

Therefore the sequence $\{x_k^*\}$ is fundamental and, because of the completeness of the space H and closeness of V, it has the limit $x^{**} \in V$.

Taking into account that the operator $P\tilde{T}x$ continuously depends on x and α_0, we can find $x^{**} = \lim_{k\to\infty} x_k^* = \lim_{k\to\infty} P\tilde{T}x_k^* = PTx^{**}$, that is, $x^{**} \in VV_0$. From the condition $2x^{**}2 \le 2x^*2$ and uniqueness of the point x^* we can conclude that $x^{**} = P_{VV_0}0$. Thus, the theorem is proved.

4. Solution of Eq. (1) by the method of increased dimension

We have reduced solution of Eq. (1) at the restriction (2) and (4) to the problem of finding the common point of V and V_1 determined by the conditions (2)–(4). Let us take the point of the minimum of the regularized functional

$$J_1\,[(x, y), (x_1, y_1)] = \alpha_0\,(\langle x, x\rangle + \langle y, y\rangle) +$$

$$+ \langle x - x_1, x - x_1\rangle + \langle y - y_1, y - y_1\rangle,$$

where $(x, y) \in V$ and $(x_1, y_1) \in V_1$, as an approximate solution.

Determine the projection operators on these sets. The projection onto V_1 $(x_1, y_1) = P_1(x, y)$ is the point of minimum in the problem

$$\min_{(x',y')\in V_1} 2(x, y) - (x', y')2$$

and it is determined by the conditions

$$x_1(t_1, t_2) = \max\,[x(t_1, t_2), 0] \text{ at } (t_1, t_2) \in \omega_0, \quad (8)$$

$$x_1(t_1, t_2) = 0 \qquad \text{at } (t_1, t_2) \notin \omega_0;$$

$$y_1(t_1,t_2) = \tilde{y}(t_1,t_2) + \frac{y - \tilde{y}}{\|y - \tilde{y}\|_\omega}\delta \quad \text{at } (t_1, t_2) \in \omega \text{ and } \| y - \tilde{y}\| > \delta;$$

$$y_1(t_1, t_2) = \tilde{y}_1(t_1, t_2) \qquad \text{at } (t_1, t_2) \in \omega \text{ and } \| y - \tilde{y}\| \le \delta; \quad (8a)$$

$$y_1(t_1, t_2) = \tilde{y}_1(t_1, t_2) \qquad \text{at } (t_1, t_2) \notin \omega.$$

The projection onto the set V $(x, y) = P(x_1, y_1)$ is the point of minimum in the problem

$$\min_{(x',y')\in V}\left(\|x'-x_1\|^2_{L_2}+\|y'-y_1\|^2_{L_2}\right)=$$
$$=\min_{x'\in H}\left(\|x'-x_1\|^2_{L_2}+\|Ax'-y_1\|^2_{L_2}\right).$$

The first coordinate x of the point of minimum satisfies the Euler equation

$$x - x_1 + A^* (Ax - y_1) = 0,$$

and it is equal to $(I + A^*A)^{-1} (x_1 + A^*y_1)$, where A^* is the operator conjugate with A.

The second coordinate of the point of minimum is $y = Ax$.

Since the operator $Ax = h(t_1, t_2)^* x(t_1, t_2)$, the conjugate operator $A^*x = h(-t_1, -t_2)^* x(t_1, t_2)$, therefore

$$Wx = (I + A^*A)^{-1} x = w(t_1, t_2)^* x(t_1, t_2),$$

where the function $w(t_1, t_2) = F^{-1} [1/(|F(h)|^2 + 1)]$. Thus,

$$(x, y) = P(x_1, y_1) =$$
$$= [W(x_1 + A^*y_1), AW(x_1 + A^*y_1)]. \quad (9)$$

According to Section 3, the iteration scheme for finding the approximate solution takes the following form:

$(x_0, y_0) \in V$ is the zero approximation,

$$(x_{n+1}, y_{n+1}) = P\left[I+\lambda\left(\frac{1}{1+\alpha_0}P_1 - I\right)\right](x_n, y_n) =$$
$$= \left[(1-\lambda)I + \frac{\lambda}{1+\alpha_0}PP_1\right](x_n, y_n), \quad (10)$$

where the operators P_1 and P are determined by Eqs. (8), (8a), and (9).

It is seen from Eq. (8a) that the operator P_1 continuously depends on $\tilde{y}$, therefore the operator $P\tilde{T}$ in the right-hand side of Eq. (10) is also continuous with respect to $\tilde{y}$. The operator $P\tilde{T}$ has the contraction coefficient $\left(1-\frac{\lambda\alpha_0}{1+\alpha_0}\right)$, which is independent of the sets V and V_1 and, consequently, of $\tilde{y}$. Therefore,[8] the immovable point of the operator $P\tilde{T}$ continuously depends on $\tilde{y}$. Thus, the solution $(x^*(\alpha_0), y^*(\alpha_0))$ found by the iteration scheme (10) is the regularized solution of Eq. (1).

5. Formal deduction of the iteration scheme (10)

Nonregularized iteration scheme (10) can be formally derived from Eq. (1). For this purpose, Eq. (1) is multiplied by A^* from the left and the resulting equation is represented in the equivalent form

$$x + A^*Ax = x + A^*y. \quad (11)$$

Equation (11) is multiplied by the operator W from the left and the resulting equation is

$$x = (1 - \lambda)\, x + \lambda\, W(x + A^*y).$$

If x and y satisfy the restriction V_1, then $(x, y) = (x_1, y_1) = P_1(x, y)$ and

$$x = (1 - \lambda)\, x + \lambda W(x_1 + A^*y_1). \quad (12)$$

Now let us write Eq. (1) in a different, but equivalent form:

$$y = (1 - \lambda)\, y + \lambda\, Ax. \quad (13)$$

If in Eq. (10) we assume $\alpha_0 = 0$, then we can see how the non-regularized iteration scheme (10) can be obtained from Eqs. (12) and (13).

The advantage of the method of increased dimension is that it allows us to point out the values of the parameter λ and the method of regularization of the iteration scheme. Besides, selection of λ and α_0 has a clear meaning.

6. Discussion and generalization

If the restriction is determined by the sets $V_1, \ldots, V_m$, rather than by a sole set V_1, then the operator P_1 in the scheme (10) should be replaced by the operator $\overline{P} = \sum_{s=1}^{m} \alpha_s P_s$. With this approach, we can take into account the property of solution smoothness. For this purpose, the solution (x, y) should be considered at the direct product of possibly different Hilbert spaces $H_1 \times H_2$ with the scalar product $\langle x_1, x_2\rangle_{H_1} + \langle y_1, y_2\rangle_{H_2}$.

If the optical system is not isoplanatic, then the Fourier method is inapplicable to finding the operator W. Therefore, it makes no sense to consider solution of Eq. (1) on the whole plane, but it is sufficient to believe that $(x, y) \in L_2(\omega_0) \times L_2(\omega)$. In this case, the algorithm (10) leads to the regularized solution with restrictions for the general integral equation of the first-kind. The considered approach to source reconstruction is also applicable, when the image is measured at discrete points and the integral in Eq. (1) is replaced with the integral sum (the case of a discrete model).

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