OSCILLATIONS OF THE MEAN LEVEL POPULATION DIFFERENCE AND ABSORPTION OF A PROBE FIELD IN A TWO–LEVEL MEDIUM IN A STRONG FIELD WITH PERIODIC AMPLITUDE MODULATION

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In this paper we present analytical solutions of the Bloch equations for a two–level medium excited by a strong resonant field with periodic amplitude modulation obtained by the technique of matrix exponent that obviates the need for operation on continued fractions. Using this approach we calculated the coefficients of absorption of a probe field in two particular cases of bichromatic and trichromatic pumping. It is shown that in contrast to trichromatic pumping the spectrum of the coefficient of absorption of the probe field for bichromatic pumping has no Rabi resonances while exhibits resonances at frequencies multiple of the intermode spacing of the pump field. Conditions for initiation and frequency regions of negative absorption (amplification) of the probe field are determined in this paper for both types of pumping.

General solutions of the Bloch equations which describe resonance interaction of a two–level medium with a polychromatic field (i.e., the field whose amplitude is periodically modulated in time) were obtained by Toptygina and Fradkin[1] using the Floquet theorem. This method was used for solving the Bloch equations with periodic coefficients in 1982 (see Ref. 2). Along with indisputable advantages, it has two substantial disadvantages: first, it can be used only for equations with periodic coefficients, second, to calculate the amplitudes of harmonics in terms of which the solutions are expressed, it is necessary to add numerically weakly convergent continued fractions.

The matrix method for solving the Bloch equations[3] enables one to obtain analytic solutions not only for periodic but also for arbitrary type of modulation of the exciting field parameters.[4] Moreover, this method obviates the need for the summation of continued fractions.

In this paper we give analytic solutions of the Bloch equations for resonance excitation of a two–level medium by a field consisting of two or three strong monochromatic components (a bi– or trichromatic field, respectively). Then we calculate the absorption coefficient of a weak field which probes into the transition saturated upon exposure to the aforementioned fields and determine frequency regions in which the absorption of the probe field is negative, i.e., there occurs its amplification.

So, let a two–level medium (a system of two–level atoms) be excited by a field whose electric component can be written in the form

$$\varepsilon(t) = E(t) \cos \omega t , \qquad (1)$$

where $E(t)$ is the amplitude which periodically depends on time and ω is the angular frequency being equal to that of transition between the levels. The equations describing time dependence of the density matrix elements of the two–level medium in the dipole–interaction approximation for the rotary wave (the Bloch equations) have the form[5]

$$\frac{dX(t)}{dt} = A(t)\, X(t) + L , \qquad (2)$$

$$X(t) = \begin{pmatrix} u(t) \\ v(t) \\ n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} -G_2 & 0 & 0 \\ 0 & -G_2 & f(t) \\ 0 & -f(t) & -G_1 \end{pmatrix},$$

$$L = n_0 \Gamma_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad f(t) = dE(t)/\hbar ,$$

where $u(t)$, $v(t)$, and $n(t)$ are the standard components of the Bloch vector in the rotating system of coordinates ($u(t)$ and $v(t)$ determine polarization of the medium in the external field and $n(t)$ determines the difference between populations of atomic levels); n_0 is the equilibrium level population difference; d is the dipole moment of transition between the levels of a two–level atom; and, $\Gamma^{-1}_{1,2}$ are the times of relaxation of population and polarization of the medium, respectively.

In some particular cases the dependence of the envelope $E(t)$ of the solution of the system of equations (2) can be obtained in an analytical form. For example, for

$$E(t) = E_0 (1 + 2 \cos \Omega t) \qquad (3)$$

field (1) can be represented as a sum of three monochromatic components with the same frequency shifts Ω between the neighboring components and the frequency of the central component ω (the so–called trichromatic field). The case

$$E(t) = 2 E_0 \cos \Omega t \qquad (4)$$

corresponds to the so–called bichromatic field (Eqs. (4) and (3) differ only in the absence of the central resonance component).

Substituting Eq. (3) in the solution of the Bloch equations for arbitrary amplitude $E(t)$ and assuming, for simplicity of calculations, the equality $\Gamma_1 = \Gamma_2 = \Gamma$, we obtain analytic solutions in an explicit form for a trichromatic exciting field

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$u(t) = 0;$

$$v(t) = n_0\, e^{-\Gamma t} \Bigg\{ \Bigg(1 - \Gamma \sum_{k=0}^{\infty} [c_k J_{2k} (P_1^-(2k) + P_1^+(2k)) -$$

$$- J_{2k+1} (P_1^-(2k+1) - P_1^+(2k+1))] \Bigg) \sin F(t) +$$

$$+ \Gamma \sum_{k=0}^{\infty} [c_k J_{2k} (P_2^-(2k) + P_2^+(2k)) -$$

$$- J_{2k+1} (P_2^-(2k+1) - P_2^+(2k+1))] \cos F(t) \Bigg\} +$$

$$+ n_0 \Gamma \sum_{k,\, l=0}^{\infty} \Big\{ c_k c_l J_{2k} J_{2l} (P_1^-(2k) - P_1^+(2k)) \sin 2(k+l)\Omega t +$$

$$+ c_k c_l J_{2k} J_{2l} (P_2^-(2k) - P_2^+(2k)) \cos 2(k+l)\Omega t +$$

$$+ [c_k c_l J_{2k} J_{2l} (P_1^-(2k) - P_1^+(2k)) +$$

$$+ J_{2k+1} J_{2l+1} (P_1^-(2k+1) - P_1^+(2k+1))] \sin 2(k-l)\Omega t +$$

$$+ [c_k c_l J_{2k} J_{2l} (P_2^-(2k) - P_2^+(2k)) +$$

$$+ J_{2k+1} J_{2l+1} (P_2^-(2k+1) + P_2^+(2k+1))] \cos 2(k-l)\Omega t +$$

$$+ [c_k J_{2k} J_{2l+1} (P_1^-(2k) + P_1^+(2k)) -$$

$$- c_l J_{2l} J_{2k+1} (P_1^-(2k+1) + P_1^+(2k+1))] \sin(2l+1+2k)\Omega t -$$

$$- [c_l J_{2l} J_{2k+1} (P_2^-(2k+1) - P_2^+(2k+1)) -$$

$$- c_k J_{2k} J_{2l+1} (P_2^-(2k) + P_2^+(2k))] \cos(2l+1+2k)\Omega t +$$

$$+ c_k J_{2k} J_{2l+1} (P_1^-(2k) + P_1^+(2k)) \sin(2l+1-2k)\Omega t -$$

$$- c_k J_{2k} J_{2l+1} (P_2^-(2k) + P_2^+(2k)) \cos(2l+1-2k)\Omega t -$$

$$- J_{2k+1} J_{2l+1} (P_1^-(2k+1) - P_1^+(2k+1)) \sin 2(k+l+1)\Omega t -$$

$$- J_{2k+1} J_{2l+1} (P_2^-(2k+1) + P(P_2^+(2k+1)) \cos 2(k+l+1)\Omega t \Big\}, \quad (5)$$

$$n(t) = n_0\, e^{-\Gamma t} \Bigg\{ \Gamma \sum_{k=0}^{\infty} [c_k J_{2k} (P_2^-(2k) + P_2^+(2k)) -$$

$$- J_{2k+1} (P_2^-(2k+1) - P_2^+(2k+1))] \sin F(t) +$$

$$+ \Bigg(1 - \Gamma \sum_{k=0}^{\infty} [c_k J_{2k} (P_1^-(2k) + P_1^+(2k)) -$$

$$- J_{2k+1} (P_1^-(2k+1) - P_1^+(2k+1))] \Bigg) \cos F(t) \Bigg\} +$$

$$+ n_0 \Gamma \sum_{k,l=0}^{\infty} \{ -c_k c_l J_{2k} J_{2l} (P_2^-(2k) - P_2^+(2k)) \sin 2(k+l)\Omega t +$$

$$+ c_k c_l J_{2k} J_{2l} (P_1^-(2k) + P_1^+(2k)) \cos 2(k+l)\Omega t -$$

$$- [J_{2k+1} J_{2l+1} (P_2^-(2k+1) - P_2^+(2k+1)) -$$

$$- c_k c_l J_{2k} J_{2l} (P_2^-(2k) - P_2^+(2k))] \sin 2(k-l)\Omega t +$$

$$+ [J_{2k+1} J_{2l+1} (P_1^-(2k+1) + P_1^+(2k+1)) +$$

$$+ c_k c_l J_{2k} J_{2l} (P_1^-(2k) + P_1^+(2k))] \cos 2(k-l)\Omega t +$$

$$+ [c_l J_{2l} J_{2k+1} (P_2^-(2k+1) + P_2^+(2k+1)) -$$

$$- c_k J_{2k} J_{2l+1} (P_2^-(2k) + P_2^+(2k))] \sin(2l+1+2k)\Omega t +$$

$$+ [c_k J_{2k} J_{2l+1} (P_1^-(2k) - P_1^+(2k)) -$$

$$- c_l J_{2l} J_{2k+1} (P_1^-(2k+1) - P_1^+(2k+1))] \cos(2l+1+2k)\Omega t -$$

$$- c_k J_{2k} J_{2l+1} (P_2^-(2k) + P_2^+(2k)) \sin(2l+1-2k)\Omega t -$$

$$- c_k J_{2k} J_{2l+1} (P_1^-(2k) - P_1^+(2k)) \cos(2l+1-2k)\Omega t +$$

$$+ c_l J_{2l} J_{2k+1} (P_2^-(2k+1) + P_2^+(2k+1)) \sin(2k+1-2l)\Omega t -$$

$$- c_l J_{2l} J_{2k+1} (P_1^-(2k+1) - P_1^+(2k+1)) \cos(2k+1-2l)\Omega t +$$

$$+ J_{2k+1} J_{2l+1} (P_2^-(2k+1) - P_2^+(2k+1)) \sin 2(k+l+1)\Omega t -$$

$$- J_{2k+1} J_{2l+1} (P_1^-(2k+1) + P_1^+(2k+1)) \cos 2(k+l+1)\Omega t \}, \quad (6)$$

where

$$P_1^{\pm}(m) = \Gamma / [\Gamma^2 + (\Omega_R \pm m\Omega)^2]\,,$$

$$P_2^{\pm}(m) = (\Omega_R \pm m\Omega) / [\Gamma^2 + (\Omega_R \pm m\Omega)^2]\,, \quad (7)$$

$\Omega_R = dE_0/\hbar$ is the Rabi frequency, $F(t) = \Omega_R t + 2\rho \sin\Omega t$, $\rho = \Omega_R/\Omega$, $J_m \equiv J_m(2\rho)$ is the Bessel function of the first kind (for simplicity we omit the argument 2ρ which is common for all J_m), $c_0 = 1/2$, $c_1 = c_2 = \ldots = c_m = 1$.

For bichromatic exciting field (4) the solutions have the form

$u(t) = 0;$

$$v(t) = n_0\, e^{-\Gamma t} \Bigg\{ \Bigg[1 - 2\Gamma \sum_{k=0}^{\infty} c_k J_{2k} L_1(2k) \Bigg] \times$$

$$\times 2 \sum_{l=0}^{\infty} J_{2l+1} \sin(2l+1)\Omega t +$$

$$+ 2\Gamma \sum_{k,\, l=0}^{\infty} c_l J_{2l} J_{2k+1} L_2(2k+1) \cos 2 l \Omega t \Big\} +$$

$$+ 2 n_0 \Gamma \sum_{k,\, l=0}^{\infty} \{[c_k J_{2k} J_{2l+1} L_1(2k) - c_l J_{2l} J_{2k+1} L_1(2k+1)] \times$$

$$\times \sin(2k+1+2l)\Omega t + [c_k J_{2l} J_{2k+1} L_2(2k+1) -$$

$$- c_k J_{2k} J_{2l+1} L_2(2k)] \cos(2k+1+2l)\Omega t -$$

$$- c_l J_{2l} J_{2k+1} L_1(2k+1) \sin(2k+1-2l)\Omega t +$$

$$+ c_l J_{2l} J_{2k+1} L_2 \cos(2k+1-2l)\Omega t +$$

$$+ c_k J_{2k} J_{2l+1} L_1(2k) \sin(2l+1-2k)\Omega t +$$

$$+ c_k J_{2k} J_{2l+1} L_2(2k) \cos(2l+1-2k)\Omega t\}, \quad (8)$$

$$n(t) = n_0 e^{-\Gamma t} \Big\{\Big[1 - 2\Gamma \sum_{k=0}^{\infty} c_k J_{2k} L_1(2k)\Big] \times$$

$$\times 2 \sum_{l=0}^{\infty} c_l J_{2l} \cos 2 l \Omega t +$$

$$+ 2\Gamma \sum_{k,\, l=0}^{\infty} J_{2k+1} J_{2l+1} L_2(2k+1) \sin(2l+1)\Omega t \Big\} +$$

$$+ 2 n_0 \Gamma \sum_{k,\, l=0}^{\infty} \{c_k c_l J_{2k} J_{2l} L_2(2k) \sin 2(k+l)\Omega t +$$

$$+ c_k c_l J_{2k} J_{2l} L_1(2k) \cos 2(k+l)\Omega t +$$

$$+ [c_k c_l J_{2k} J_{2l} L_2(2k) + J_{2k+1} J_{2l+1} L_2(2k+1)] \times$$

$$\times \sin 2(k-l)\Omega t + [c_k c_l J_{2k} J_{2l} L_1(2k) +$$

$$+ J_{2k+1} J_{2l+1} L_1(2k+1)] \cos 2(k-l)\Omega t -$$

$$- J_{2k+1} J_{2l+1} L_2(2k+1) \sin 2(k+1+l)\Omega t -$$

$$- J_{2k+1} J_{2l+1} L_1(2k+1) \cos 2(k+1+l)\Omega t\}, \quad (9)$$

where

$$L_1(m) = \Gamma/(\Gamma^2 + m^2\Omega^2), \; L_2(m) = m\Omega/(\Gamma^2 + m^2\Omega^2). \quad (10)$$

It should be noted here that the analogous solution for a bichromatic exciting field (the case of the so–called fully amplitude–modulated excitation) in the form of expansion in the Bessel functions was obtained in Ref. 8. However, these solutions can be represented in an explicit form only for two limiting cases: very weak and very strong excitating fields when one can neglect relaxation in comparison with Rabi frequency (i.e., in the approximation $\Gamma \ll \Omega_R$). Our solutions (8) and (9) are valid for arbitrary intensity of exciting field and for $\Gamma \to 0$ they coincide with the solutions derived in Ref. 8.

By comparing solutions (5)–(6) and (8)–(9), it is readily seen that the spectrum of undamped oscillations of the Bloch vector components for bichromatic excitation contains an infinite set of harmonics at the intermode frequency Ω. For $n(t)$ it is a set of even and for $\nu(t)$ a set of odd harmonics. For a trichromatic exciting field, both $\nu(t)$ and $n(t)$ contain an infinite set of even and odd harmonics Ω. One more difference between trichromatic and bichromatic excitation is as follows [as seen from Eq. (7)]. In the first case the harmonic amplitudes increase resonantly when $\Omega_R = \pm m\Omega$ (the conditions of the so–called Rabi resonance), and in the case of bichromatic excitation these resonances are absent.

Figure 1 depicts a plot of the time–averaged level population difference $\overline{n}$ vs. the intermode spacing Ω which was calculated numerically from Eqs. (6) and (9). Trichromatic excitation (curve *1*), in contrast to a bichromatic one (curve *2*) leads to Rabi resonances in the spectrum of $\overline{n}$. Common to these solutions is that $\overline{n}(\Omega)$ does not take negative values for any value of detuning of the atomic transition frequency from the radiation frequency, i.e., it is impossible to obtain the field amplification. Such negative absorption (amplification) in a two–level medium can appear only for a weak probe field when the resonant transition is saturated upon exposure to a strong field.

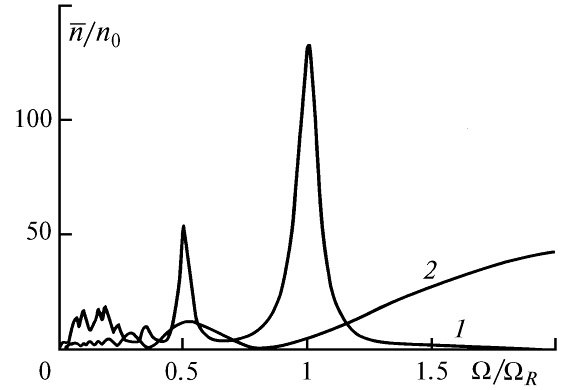

FIG. 1. Time–averaged level population difference $\overline{n}$ vs. the intermode spacing Ω for trichromatic (1) and bichromatic (2) excitation.

In Ref. 3 the analytical expression for the polarization of a medium in a weak probe field at variable frequency with simultaneous excitation of this medium by a strong trichromatic field at fixed frequencies was derived, the absorption coefficient of this probe field was calculated, and the frequency regions in which this absorption becomes negative were found. In the present paper we show that bichromatic pumping, in contrast to trichromatic one, results in the substantially different spectrum of the absorption coefficient of a probe field. For the field of the type $\varepsilon(t) = 2E_0 \cos\Omega t \cos\omega t + \varepsilon_0 \cos(\omega + \delta)t$, where $\varepsilon_0 \ll E_0$ and $|\delta| \ll \omega$ is the weak field detuning from resonance, an analytical expression for the components of the Bloch vector and hence for polarization of the medium induced by this field can readily be obtained using the method described in Ref. 3.

The derived expressions are too cumbersome, and we present here only the spectra of the absorption coefficient of the probe field obtained using these solutions for two values of the amplitude of strong bichromatic pumping: $\Omega_R = 10\Gamma$ and $\Omega_R = 20\Gamma$ (curves *1* and *2* in Fig. 2, respectively).

An imaginary part of the medium polarizability $\mathrm{Im}\,\chi(\delta)$, induced by a probe field, which is directly proportional to the absorption coefficient of the external field (see, e.g., Ref. 6, where the absorption coefficient $\kappa(\delta) = \omega\,\mathrm{Im}\,\chi(\delta)/(n\,c\,N)$, n is the refractive index of the medium, c is the light speed in vacuum, and N is the density of absorbing atoms), for trichromatic saturation of the transition, is given in Fig. 3 for $\Omega_R = 10\Gamma = \Omega$.

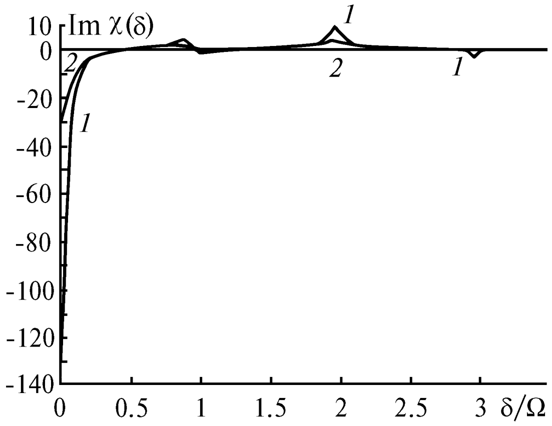

FIG. 2. Imaginary part of polarizability of a medium induced by a probe field for bichromatic pumping at $\Omega_R = 10\Gamma$ *(1) and* 20Γ *(2).*

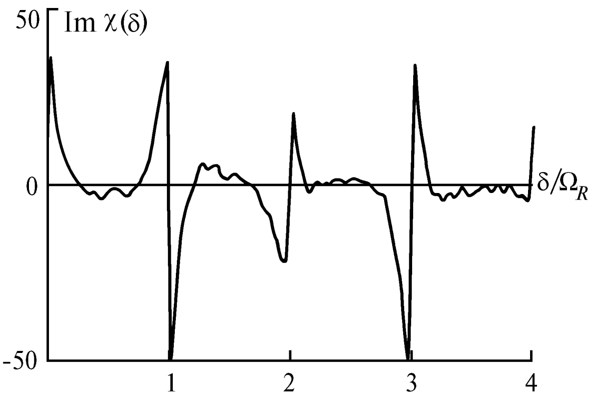

FIG. 3. Imaginary part of polarizability of a medium induced by a probe field for trichromatic pumping at $\Omega_R = 10\Gamma$.

The following conclusions can be drawn from the comparison of these plots:

1. For bichromatic pumping (without resonance component) the spectrum of an imaginary part of the medium polarizability $\mathrm{Im}\,\chi(\delta)$ determining the probe field absorption exhibits resonances in the vicinity of detunings $\delta = m\Omega$, $m = 0, \pm1, \pm2, \ldots$ (intermode resonances). In the vicinity of $\delta = \Omega$ the absorption has a dispersive character and for the remaining m the absorption curve has the Lorentz shape. In this case $\mathrm{Im}\,\chi(\delta)$ and hence the absorption are negative in the vicinity of $\delta = 0$ and $\delta = \pm\Omega$. The negative absorption reaches maximums in the vicinity of exact resonance. It should be noted that the stronger is pumping, the smaller is the value of negative absorption. It is accounted for by the fact that for exact resonance the weak field, being added to pumping, additionally promotes the transition saturation. The stronger is pumping, the closer is the medium to saturation and hence the weaker is absorption of the external field.

2. The dependence of the spectrum of an imaginary part of the polarizability of a medium induced by a probe field on detuning for trichromatic pumping is more complicated. First, the resonances appear at $\delta = m\Omega = m\Omega_R$, i.e., intermode resonances in this case coincide with Rabi resonances. It should be noted that these Rabi resonances occur when δ coincides with harmonics of the Rabi frequency Ω_R, while the Rabi resonances in the components of the Bloch vector of a medium excited by a trichromatic field (without a probe field) are observed when the intermode frequency Ω coincides with subharmonics of the frequency Ω_R (see Ref. 7). The spectrum has a dispersive character in the vicinity of all resonance values of δ (as it should be for strong trichromatic pumping[7]). The regions of negative $\mathrm{Im}\,\chi(\delta)$, i.e., regions of negative absorption become pronounced in the vicinities of the Rabi resonances. In the case of exact resonance ($\delta = 0$) a weak field is added to a resonance component of pumping thereby also promoting the transition saturation. The negative absorption is lacking in this case. The widths of all resonance lines are the same and exceed the widths of lines for trichromatic pumping with the same values of the parameters.

When detunings $\delta > 3\Omega$, probe field no longer induces transitions between the levels, and the system is in the state of saturation caused by strong pumping.

Thus the maximum amplification of the field with periodically modulated amplitude should occur in the case of saturation of a two–level medium (e.g., in a laser cavity) in a polychromatic field when the frequency of the transition between the levels of the medium lies in the middle between the neighboring spectral components of the field.

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