RECONSTRUCTION OF AN OPTICAL SIGNAL BY THE CONVEX ANALYSIS METHODS

S.M. Chernyavskii

A.N. Tupolev State Technical University, Kazan'
Received July 7, 1995

Two inverse problems of optics are examined by the convex analysis methods: reconstruction of a phase component of an optical signal from the known module and from the module of its Fourier transform within a given region as well as the reconstruction of incoherent source from its noise disturbed image. The solution of each problem is reduced to seeking a boundary point of a convex set with a given extreme property.

Reconstruction of an optical signal from its image is related to solution of ill-posed problems. Restriction of the solution set is one of the methods of its regularization. The methods of seeking solutions of inverse problems on a given convex sets was found to be efficient and many papers develop this approach. For instance, among them is a paper by D.C. Youla[1] in which the problem of signal reconstruction is considered as a problem of seeking the intersection point of given convex sets and is solved by an iterative method of a successive projection onto the sets. An iterative algorithm of finding the radiation source from its noise disturbed image under the condition that the values of the radiant flux and noise are within limits of the given convex sets is considered in Ref. 2. In the present paper we consider the method of signal reconstruction on a convex set with a given extremal property. This can be expressed in a dual form by the condition that the given point cannot be separated from the convex set by a hyperplane. The dual condition enables one to study the properties of the solution sought and to construct approximate methods of finding it. In particular, Tikhonov's regularization[3] of a solution by finding the solution with the minimum norm when the value of the residual norm is given can be considered as a problem of finding a given boundary point on a convex set.

RECONSTRUCTION OF THE PUPIL FUNCTION PHASE FROM A KNOWN POINT SCATTERING FUNCTION IN A GIVEN DOMAIN

The pupil function $G(\xi, \eta) = A(\xi, \eta)\exp(i\Phi(\xi, \eta))$ has its domain Ω in the plane $0\xi\eta$. The amplitude $A(\xi, \eta)$ and the point scattering function $h(x, y) = |g(x, y)|^2$ in the domain ω of the image plane $0xy$, where $g = F(G)$ is the Fourier transform of G, are assumed to be known. The phase function $\Phi(\xi, \eta)$ is to be found under these conditions. By a solution of the phase problem we mean finding of a function satisfying the condition

$$|F(G)|^2 = h(x, y) \quad \text{for} \quad (x, y) \in \omega. \tag{1}$$

Let us set two norms on the set of functions in the $0xy$ plane

$$\|g\|^2 = \int\limits_{-\infty}^{\infty}\!\!\int |g(x, y)|^2 \mathrm{d}x\, \mathrm{d}y$$

and

$$J(g) = [(\|g\|_{2/p})^2 + (\|g\|_2)^2]^{1/2},$$

where the semi-norms

$$(\|g\|_{2/p})^{2/p} = \iint\limits_{\omega} |g(x,y)|^{2/p}\, \mathrm{d}\mu(x, y),$$

$$(\|g\|_2)^2 = \iint\limits_{\omega} |g(x, y)|^2\, \mathrm{d}x\, \mathrm{d}y,$$

$$\omega + \omega' = 0xy, \quad 2/p > 1, \quad p^{-1} + q^{-1} = 1,$$

$$\mathrm{d}\mu = \rho(x, y)\mathrm{d}x\, \mathrm{d}y, \quad \rho = h^{1/q}(x, y)/M^{1/q},$$
$$M = \iint\limits_{\omega} h(x, y)\mathrm{d}x\, \mathrm{d}y.$$

The following inequality that can easily be checked is very important for the below discussion

$$\|g\| \geq J(g). \tag{2}$$

L e m m a 1. The inequality (2) becomes an equality if and only if the functions g satisfy the condition

$$|g(x,y)|^2 = ch(x, y), \quad c > 0, \quad (x, y) \in \omega, \tag{3}$$

Sufficiency of the condition (3) can be proved directly by substitution into the inequality (2). Necessity: By the definition of the norms $\|g\|$ and $J(g)$ the inequality (2) becomes an equality for functions g for which

$$\min_g \left[\iint_\omega |g(x, y)|^2 \mathrm{d}x\, \mathrm{d}y - \right.$$

$$\left. - \left(\iint_\omega |g(x, y)|^{2/p} \mathrm{d}\mu(x, y) \right)^p \right] = 0.$$

Calculating the derivative of the functional and equating it to zero we find that

$$2|g| - p(\|g\|_{2/p})^{2/q}\, 2p^{-1} \|g\|^{2/p-1} \rho = 0$$

Setting $g \neq 0$ we obtain that $|g(x,y)|^2 = h(x, y)(\|g\|_{2/p})^2/M$ what is equivalent to Eq. (3).

Let B_1 and B_2 denote two Banach function spaces on the plane $0xy$ with the norms $\|g\|$ and $J(g)$, respectively. The inequality (2) implies that $B_1 \subset B_2$. For the function $g = F(G) \in B_1$ the Plancherel's equality is valid

$$\|g\|^2 = \iint_\Omega |G(\xi, \eta)|^2 \mathrm{d}\xi\, \mathrm{d}\eta =$$

$$= \iint_\Omega A^2(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = l^2,$$

where the value l is known because the amplitude A is known.

Let S_1 and S_2 be closed spheres of radius l in the spaces B_1 and B_2, ∂S_1 and ∂S_2 be their boundaries. From inequality (2) and Lemma 1 we conclude that $S_1 \subset S_2$ and $\partial S_1 \cap \partial S_2 \neq \varnothing$. Thus the sphere S_1 is an oblate formation inside the sphere S_2 and it is extended in the direction of the boundary points at which the norm does not change under the identity transformation from the points of the space B_1 into those of B_2.

Let us consider a closed limited set in B_1 (accessibility set):

$$V = \{g\colon g = F(G),\ |G(\xi, \eta)| \leq A(\xi, \eta)\} \subset S_1 \subset S_2,$$

for which the following lemma is valid.

L e m m a 2 . If the phase problem has a solution satisfying the condition (3), there exists a solution satisfying the condition (1); the condition

$$V \cap \partial S_2 \neq \varnothing \tag{4}$$

is necessary. If ω coincides with $0xy$, then $c = 1$ in Eq. (3).

Let us write the condition (4) in a dual form. A linear functional λ from the conjugate space B_2^* can be defined by the equality

$$\lambda(g) = \iint_\omega g(x, y)\lambda^*(x, y)\mathrm{d}\mu(x, y) +$$

$$+ \iint_{\omega'} g(x, y)\lambda^*(x, y)\mathrm{d}x\mathrm{d}y,$$

where asterisk denotes complex conjugation, and it has the norm

$$\left((\|g\|_{2/q'})^2 + (\|g\|_2)^2 \right)^{1/2}, \quad q' + p = 2.$$

The inclusion $V \subset S_2$ is equivalent to the inequality

$$\max_{g \in V} \mathrm{Re}\lambda(g) \leq \max_{g \in S_2} \mathrm{Re}\lambda(g) = l\|\lambda\| \quad \text{for all } \lambda \in B_2^*,$$

which is equivalent to the inequality

$$\sup_{\|\lambda\|=1} \max_{g \in V} \mathrm{Re}\lambda(g) \leq l \tag{5}$$

because of its homogeneity. For each $g_0 \in B_2$ the inequality $\mathrm{Re}\ \lambda(g) \leq J(g)\ \|\lambda\|$ is valid for all λ, but there exists a unique (extremal) functional λ_0 for which $\|\lambda\| = 1$ and $\mathrm{Re}\ \lambda_0(g_0) = \lambda_0(g_0) = J(g_0)\|\lambda_0\| = J(g_0)$. The extremal functional of the function $g = |g(x, y)|\exp(i\varphi(x, y))$ has the form

$$\lambda(x,y) = J^{-1}(g)(\|g\|_{2/p})^{2/q'}|g(x,y)|^{q'/p}\exp(i\varphi(x,y))$$

for $(x, y) \in \omega$

and

$$\lambda(x, y) = J^{-1}(g)\ |g(x, y)|\exp(i\varphi(x, y))$$

for $(x, y) \in \omega'$.

Let $g_0 \in V \cap \partial S_2$ and λ_0 be the extremal functional of g_0, then

$$\mathrm{Re}\ \lambda_0(g_0) = J(g_0) = l. \tag{6}$$

The expressions (5) and (6) imply that the function $g \in V$ with the maximum norm $J(g) = l$ is a solution of the extremal problem

$$\max_{\|\lambda\|=1} \max_{g \in V} \mathrm{Re}\ \lambda(g) = l. \tag{7}$$

The expression $\lambda(F(G)) = \Lambda(G)$ defines a linear functional Λ in the space L_2 on the plane $0\xi\eta$. Here $\Lambda = F^{-1}(\lambda\rho')$, where $\rho' = \rho$ on ω and $\rho' = 1$ on ω'. If one introduces a set $U = \{G\colon |G(\xi, \eta)| \leq A(\xi, \eta)\}$, the equation (7) can be written in the form

$$\max_{\|\lambda\|=1} \max_{G \in U} \mathrm{Re}\Lambda(G) = l. \tag{8}$$

If λ^0 and G^0 make a solution to the problem (8) and $\Lambda^0 = F^{-1}(\lambda^0\rho')$, then G^0 satisfies the maximum condition

$$\mathrm{Re}\ \Lambda^0(G^0) = \max_{G \in U} \mathrm{Re}\ \Lambda^0(G) =$$

$$= \iint_\Omega A(\xi, \eta)\Lambda^0(\xi, \eta)\mathrm{d}\xi\, \mathrm{d}\eta\ ,$$

where

$$G^0 = A(\xi, \eta)\Lambda^0(\xi, \eta)/|\Lambda^0(\xi, \eta)|\ . \tag{9}$$

The function G^0 satisfies the condition $|G^0(\xi, \eta)| = A(\xi, \eta)$ so it can be considered as a permissible pupil function; the condition (4) and, hence, (3) are fulfilled for it. Moreover, if $c = 1$, G^0 is the solution of the phase problem.

METHODS OF CONSTRUCTING A MAXIMIZING SEQUENCE

1. It is necessary to construct a maximizing sequence λ_k, $k = 0, 1, 2,\ldots$,

$$\|\lambda_k\| = 1, \quad f(\lambda_k) < f(\lambda_{k+1}), \quad \lim_{k\to\infty} f(\lambda_k) = l.$$

in the problem

$$\max_{\|\lambda\|=1} f(\lambda) = l,$$

where

$$f(\lambda) = \max_{g\in V} \mathrm{Re}\, \lambda(g), \tag{10}$$

Let G_k be a permissible pupil function, $g_k = F(G_k)$, λ_k be the extremal functional of g_k, $\Lambda_k = F^{-1}(\lambda_k \rho')$, G_{k+1} be a function satisfying the maximum condition (9) on λ_k and $g_{k+1} = F(G_{k+1})$, λ_{k+1} be the extremal functional of g_{k+1}. The chain of inequalities $J(g_k) = \mathrm{Re}\,\lambda_k\,(g_k) \le f(\lambda_k) = \mathrm{Re}\,\lambda_k(g_{k+1}) \le J(g_{k+1}) = \mathrm{Re}\lambda_{k+1}(g_{k+1}) \le f(\lambda_{k+1})$ is valid. If $g_k \ne g_{k+1}$, the second inequality is strict due to the uniqueness of the extremal functional for g, so $J(g_k) < J(g_{k+1})$ and $f(\lambda_k) < f(\lambda_{k+1})$. Two cases are possible here: 1) λ_k is the maximizing sequence; 2) $g_k = g_{k+1}$ for a certain k. Moreover, if $f(\lambda_k) = l$, λ_k is the solution to the problem (10). If $f(\lambda_k) \ne l$, one should test new $g \in V$ with a larger norm as compared with $J(g_k)$.

2. Method of alternating projections. Let g be a point and V be a closed set in B_2. A point $g_1 \in V$ is said to be a projection of g onto V if

$$J(g_1 - g) = \min_{g'\in V} J(g' - g), \tag{11}$$

and is denoted by $g_1 = P_V\, g$, where P_V denotes the operation of projection (11).

In accordance with Lemma 2 the phase problem is reduced to seeking the point $g \in V \cap S_2$. The sequence maximizing the norm is defined by the iterative relation

$$g_k \in V, \quad g_{k+1} = P_V\, P_{S_2} g_k, \quad k = 0, 1, 2, \ldots,$$
$$P_{S_2} g = lg/J(g), \tag{12}$$

where $P_V\, g$ is the solution of the convex programming problem (11).

The following three relationships

$$J(P_{S_2} g_k - g_k) \ge J(P_{S_2} g_k - g_{k+1}), \tag{13}$$

$$J(P_{S_2} g_k) = J(P_{S_2} g_k - g_k) + J(g_k),$$

$$J(P_{S_2} g_k) \le J(P_{S_2} g_k - g_{k+1}) + J(g_{k+1}).$$

are valid by definitions of the projection, operator P_{S_2}, and the triangle axiom for norms. From the 2nd and the 3rd relations we obtain that

$$J(g_k) + [J(P_{S_2} g_k - g_k) - J(P_{S_2} g_k - g_{k+1})] \le J(g_{k+1}). \tag{14}$$

Two cases are possible:

1. $g_k \ne g_{k+1}$. Then, because of the uniqueness of the projection in B_2, there is a strict inequality in Eq. (13). Taking this into account, we have $J(g_k) < J(g_{k+1})$ from the expression (14). If $J(g_k) \to l$, the sequence (12) is maximizing.

2. For a certain k $g_k = g_{k+1}$. If $J(g_k) = l$, g_k is the solution. Otherwise, g_k is an intermediate "vertex" of the set V. It is necessary to seek a new point g_{k+1} such that $J(g_{k+1}) > J(g_k)$.

RECONSTRUCTION OF AN INCOHERENT RADIATION SOURCE FROM A GIVEN POINT SCATTERING FUNCTION $h(x, y; x_0, y_0)$ AND A NOISE DISTURBED IMAGE

Let the distribution of the radiation source intensity and its noise disturbed image be described by the functions $I_0(x_0, y_0)$ and $I(x, y)$. It is necessary to reconstruct I_0 in a domain E_0 from a given I in a domain $E \subset E_0$ and from a given function $h(x, y; x_0, y_0)$.

The functions considered here are connected by the superposition integral

$$I(x, y) = \int\int_{E_0} h(x, y; x_0, y_0) I_0(x_0, y_0) \mathrm{d}x_0 \mathrm{d}v_0 + z(x, y),$$
$$(x, y) \in E, \tag{15}$$

where the function $z(x, y)$ characterizes the noise influence and solution inaccuracy of the equation (15). We assume that $I_0 \in L_{p_0}(E_0)$, and $I, z \in L_p(E, \mu)$, $p_0,\ p > 1$. The measure $\mu = \mu(I)$ depends on I and reflects the purpose of the function z in a certain sense. A pair of functions $I_0 \in L_{p_0}(E_0)$ and $z \in L_p(E, \mu)$ satisfying the equation (15) is said to be a solution of this equation. Below the equation (15) will be written in the operator form $I = hI_0 + z$.

We choose the average intensity level $I_{0\mathrm{av}}(x_0, y_0)$ so that the function $u(x_0, y_0)$ has an arbitrary sign in the expression $I_0 = I_{0\mathrm{av}} + u$. If we assume $a = I - hI_{0\mathrm{av}}$, the equation (15) takes the form

$$a = hu + z. \tag{16}$$

In order to restrict the set of possible solutions we introduce two convex closed and limited sets $U \in L_{p_0}(E_0)$ and $Z \in L_p(E, \mu)$ depending on I in the general case, $0 \in \mathrm{int}\, Z$. A solution (u, z) is said to be permissible if $u \in \alpha U$ and $z \in \delta Z$ where α and δ are parameters enabling one to deform the sets. By choosing the sets U and Z one can take into account information about the solution and noise, and make the solution unique and correct. The problem of reconstructing the source is reduced to seeking a permissible solution of the equation (16).

Let us consider the set $V = \{g: \ g = hu + z, u \in \alpha U, \ z \in \delta Z\}$ which is convex, closed, limited, and int $V \neq \varnothing$. The existence of a permissible solution to equation (16) means that a point $a \in V$ or, in equivalent dual form, for all $\lambda \in L_p(E, \mu)$ the inequality

$$\lambda(a) \le \max_{g \in V} \lambda(g) = \max_{u \in \alpha U} u(h^*\lambda) + \max_{z \in \delta Z} \lambda(z). \tag{17}$$

is valid. Here h^* is the operator conjugate to h. By choosing the parameters α^0 and δ^0 one can make the point a to be boundary for the set V. Then an equality is achieved in inequality (17) at a certain functional λ^0. Taking into account this fact and homogeneity of inequality (17) with respect to λ, one can write it in the form

$$\delta \ge \delta^0 = \max_{p(\lambda)=1} \{\lambda(a) - R(\alpha^0, \lambda)\}; \tag{18}$$
$$R(\alpha, \lambda) = \max_{u \in \alpha U} u(h^*\lambda); \quad p(\lambda) = \max_{z \in Z} \lambda(z).$$

The condition (18) is necessary and sufficient for the existence of a permissible solution to equation (16) satisfying the condition $u \in \alpha^0 U$, $z \in \delta Z$. The connection between the solution of the variational problem and that of equation (16) yields the following proposition.

P r o p o s i t i o n . If λ^0 and δ^0 make a solution to the variational problem, the functions u^0 and z^0 satisfying the maximum condition

$$u^0(h^*\lambda^0) = \max_{u \in \alpha^0 U} u(h^*\lambda^0), \quad \lambda^0(z^0) = \max_{z \in \delta^0 Z} \lambda^0(z),$$

are the solution of equation (16).

The condition for resolution of equation (16) written in the form (18) permits one to establish the conditions under which the solution of equation (16) has preset properties. Suppose that a functional $p(\lambda)$ is strictly convex. For instance, we have such a functional if $Z = Z_1 = \{ \|z\| \le 1\}$. Let us introduce a generalized parameter $B = \{a, \alpha\}$, $\|B\| = \|a\| + |\alpha|$, such that the maximized functional in Eq. (18) depends on it.

The properties of the solutions of the variational problem (18) and permissible solutions of equation (16) are defined by the following lemmas.

L e m m a 1 . $\delta^0(B)$ is a continuous function.

L e m m a 2 . If $\delta^0 > 0$, the variational problem (18) has a unique solution.

Proof. Suppose the contrary, i.e., let there exist two solutions λ_1 and λ_2. Then the equalities $\delta^0 = \lambda_i(a) - R(\alpha, \lambda_i)$, $i = 1, 2$, are valid for them. Summing these two equalities and taking into account that the functional $R(\alpha, \lambda_1)$ is convex relative to λ, we obtain

$$2\delta^0 \le (\lambda_1 + \lambda_2)(a) - R(\alpha, \lambda_1 + \lambda_2). \tag{19}$$

The left-hand side in inequality (19) is homogeneous with respect to λ and $\lambda_1 \neq \lambda_2$; otherwise, the condition $\delta^0 > 0$ is broken. Divide both sides of the inequality (19) by the number $\beta = p(\lambda_1 + \lambda_2) < p(\lambda_1) + p(\lambda_2) = 2$: $(2/\beta)\delta^0 \le ((\lambda_1 + \lambda_2)/\beta)(a) - R(\alpha, (\lambda_1 + \lambda_2)/\beta)$. Since $2/\beta > 1$ and $\lambda = (\lambda_1 + \lambda_2)/\beta$ satisfies the condition $p(\lambda) = 1$, we come to a contradiction with the definition of δ^0: $\delta^0 < \lambda(a) - R(\alpha, \lambda)$.

L e m m a 3 . If the sequence of parameters B_i converges by norm to B' and $\delta^0(B') > 0$, the sequence of solutions of the variational problem (18) $\lambda(B_i)$ has a subsequence slightly converging to $\lambda(B')'$.

Proof. The subsequence $\lambda(B_i)$ belongs to a slightly compact set $\{p(\lambda) \le 1\}$. It contains a subsequence $\lambda(B_{ik})$ slightly converging to λ'. Let us show that $\lambda' = \lambda^0(B')$. Suppose the contrary. Then, the inequality

$$[\lambda'(a) - R(\alpha, \lambda')] < \delta^0(B'). \tag{20}$$

is valid according to Lemma 2. The functional in Eqs. (18) is continuous and convex with respect to λ, so it is slightly semi-continuous from the above[4]; but then the inequality

$$\delta^0(B_{ik}) = [\lambda_{ik}(a_{ik}) - R(\alpha_{ik}, \lambda_{ik})] =$$
$$[\lambda_{ik}(a_{ik}) - R(\alpha_{ik}, \lambda_{ik})] - [\lambda_{ik}(a') - R(\alpha', \lambda_{ik})] +$$
$$+ [\lambda_{ik}(a') - R(\alpha', \lambda_{ik})] \le c \, \|a_{ik} - a'\| - [R(\alpha_{ik}, \lambda_{ik}) -$$
$$- R(\alpha', \lambda_{ik})] + [\lambda'(a') - R(\alpha', \lambda')] \le c \, \|a_{ik} - a'\| +$$
$$+ c_1 \|\alpha_{ik} - \alpha'\| + [\lambda'(a') - R(\alpha', \lambda')],$$

is valid. Proceeding to the limit we come to contradiction with inequality (20)

$$\delta^0(B') \le [\lambda'(a') - R(\alpha', \lambda')].$$

C o r o l l a r y 1. Let $Z = Z_1$. Then the solution of the variational problem (18) $\lambda^0(B)$ continuously depends on B in the domain $\{B\}$ where $\delta^0(B) > 0$. If the maximum condition defines u as a continuous function of $h^*\lambda$, then u continuously depends on B.

Proof. Slight convergence of the subsequence $\lambda(B_{ik})$ to $\lambda^0(B')$ under the condition $\|\lambda(B_{ik})\| = \|\lambda^0(B')\|$ provides the convergence by norm. But then the sequence $\lambda(B_i)$ converges to $\lambda^0(B')$ by norm because B_i is an arbitrary sequence converging to B'.

C o r o l l a r y 2. Let h be a compact operator and the maximum condition define u as a continuous function of $h^*\lambda$. Then u continuously depends on B in the domain $\{B\}$ where $\delta^0(B) > 0$.

Let us consider a maximizing sequence λ_i in the problem (18), functions u_i defined by the maximum condition at λ_i, $z_i = a - h^* u_i$, and $\delta_i = \min\delta$ for which $z_i \in \delta Z$. Let the condition of maximum define u as continuous function of $h^*\lambda$, then the following lemma is valid.

Lemma 4. If $Z = Z_1$, or h is a compact operator, the estimation made from two sides

$$\delta_i \geq \delta^0 \geq \lambda_i(a) - R(\alpha, \lambda_i),$$

in which the limiting values tend to δ^0 is valid in the domain $\{B\}$ where $\delta^0(B) > 0$.

The proofs of Lemma 1 and Lemma 4 are omitted since they are similar to those presented.

REFERENCES

1. D.K. Yula, in: *Image Recovery: Theory and Application,* ed. by H. Stark (Academic Press, INC., 1987).
2. S.I. Kavkyanov and S.V. Samoilova, in: *ABstracts of Papers at the 1st InterrepuBlic Symposium on Atmospheric and Oceanic Optics*, Tomsk (1994), pp. 120–121.
3. A.N. Tikhonov and V.Ya. Arsenin, *Methods of Solving Ill-Posed ProBlems* (Nauka, Moscow, 1974), 222 pp.
4. B.T. Polyak, Dokl. Akad. Nauk SSSR **166**, No. 2, 287–290 (1966).