FREQUENCY SUMMING IN FOCUSED BEAMS

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A new approach is proposed for solving problems of nonlinear interaction of focused beams in anisotropic crystals. The approach is based on the use of the methods of the theory of Green's functions. The solution obtained in the fixed-field approximation for the converted wave is extended to the case of the second-harmonic generation in the strongly nonlinear generation regime.

INTRODUCTION

The mathematical aspect of many problems in radio physics and optics (including the nonlinear aspect) is simplified to one or smother extent, if the starting wave equation cam be replaced by an equation of the so-called truncated parabolic type. The advantages and drawbacks of this approach are examined for some specific questions, for example, in Refs. 1–3. The great popularity of the parabolic approximation does not preclude using Maxwell's equations, especially since the mathematical difficulties connected, for example, with the calculation of the Green's tensor of an anisotropic medium[4,5] are for a number of cases not so daunting.[6]

The scheme proposed in this paper for solving the problem of nonlinear optical processes at a focal point located inside an anisotropic crystal has a fully standard form: relations of the Kirchhoff formula type (with the Green's tensor $\tilde{G}$ of the crystalline medium) give the intensity $\vec{E}_1$ of the "linear field" in a neighborhood of the focal point, and this region is the source of nonlinear waves $\vec{E}_{n1}$ (second harmonic, some frequency, etc.). In the fixed-field approximation[3] this approach leads to the expressions

$$E_{n1} \sim \int E_1 E_1 \tilde{G}\, dr.$$

The computational difficulty, that is unavoidable in the mathematically exact formulation of the problem, is in our opinion fully justified; since the application of the parabolic approximation at the points of convergence of the geometric-optics rays occasionally rises certain doubts (see the comprehensive mathematical analysis given in Ref. 1). (Of course, this question requires a special discussion and we intend to publish the corresponding analysis).

At the first stage a quite good approximation was found for the Green's tensor of an anisotropic medium and the solution of the wave equation for the monochromatic vector field in a uniform uniaxial crystal was written down.[6] Then an expression was obtained in the fixed-field approximation for the wave at the sum frequency under arbitrary boundary conditions.[7]

These results are used here to solve the problem of three-frequency interaction of focused beams in a KDP crystal with a scalar OOe synchronism. It should be noted that for KDP with the synchronism of the type chosen the expression for $\vec{E}_{n1}$ contains only the components of $\tilde{G}$ that are calculated exactly. For this reason in the region near the focal point the solution obtained on the basis on the fixed-field approximation is exact.

1. SUMMATION OF FREQUENCIES IN FOCUSED BEAMS. FIXED-FIELD APPROXIMATION

For the three-frequency interaction of waves in a quadratic dielectric we have the system of equations

$$\begin{aligned}
&\mathrm{rot\ rot}\vec{E}_1 - \tilde{\varepsilon}_1\vec{E}_1 = \tilde{\chi}_1\vec{E}_3\vec{E}_2^{*};\\
&\mathrm{rot\ rot}\vec{E}_2 - \tilde{\varepsilon}_2\vec{E}_2 = \tilde{\chi}_2\vec{E}_3\vec{E}_1^{*};\\
&\mathrm{rot\ rot}\vec{E}_3 - \tilde{\varepsilon}_3\vec{E}_3 = \tilde{\chi}_3\vec{E}_1\vec{E}_2;
\end{aligned} \tag{1.1}$$

where $\tilde{\varepsilon}_j = (\omega_j^2/c^2)\tilde{\varepsilon}(\omega_j)$, $\tilde{\chi}_j = (4\pi\omega_j^2/c^2)\tilde{\chi}_j(\omega_j)$, $\omega_3 = \omega_1 + \omega_2$, $\tilde{\varepsilon}(\omega_j)$ and $\tilde{\chi}(\omega_j)$ are the dielectric constant tensor and the quadratic nonlinear susceptibility tensor.

In the fixed-field approximation the right side of the first two equations can be neglected and the following expression is obtained for the intensity of the magnetic field at the sum frequency in the case of scalar OOe synchronism in the KDP crystal[7]

$$\vec{H}_3(\vec{r}_0) = \frac{2n_{30}\omega_3^2}{c^2}\chi_{123}(\omega_3)\times \int\limits_V E_{1y}E_2\frac{\vec{k}(y-y_0)-\vec{j}(z-z_0)}{R'^2}\cdot e^{ik_{3e}R'}d\vec{r}, \tag{1.2}$$

where E_{1y} and E_{2z} are the components of vectors satisfying the corresponding homogeneous wave equations[6]

$$n_{jo,e} = \sqrt{\varepsilon_{o,e}(\omega_j)},$$

$$k_{jo,e} = \left[\omega_j/c\right] n_{o,e}(\omega_j),$$

$$\chi_{123} = (\tilde{\chi})_{xyz}, \quad \beta = n_{3e}/n_{3o}$$

and

$$R' = \sqrt{\frac{(x - x_0)^2}{\beta^2} + (y - y_0)^2 + (z - z_0)^2}.$$

The solution Eq. (1.2) was written in the principle dielectric coordinates. The principle optical axis of the medium is oriented along the x-axis, i.e.,

$$\varepsilon_{xx} \equiv \varepsilon_e, \qquad \varepsilon_{yy} = \varepsilon_{zz} \equiv \varepsilon_o.$$

For analysis of the problem of generation of the sum frequency near the focal point of a lens (with the focal length f) the following picture was adopted:

a) the "crystal" occupies the entire space $z > 0$;

b) at $z = 0$ (in the medium) two ordinary spherical beams are given:

$$\vec{E}_1(\xi, \eta) = \vec{e}_1(\xi, \eta) \; A_1(\xi, \eta)\exp\left[-ik_{10}R\xi\right],$$

$$\vec{E}_2(\alpha, \varphi) = \vec{e}_2(\alpha, \varphi) \; A_2(\alpha, \varphi)\exp\left[-ik_{20}R_\alpha\right], \tag{1.3}$$

where

$$R_\xi = \sqrt{(x_f - \xi)^2 + (y_f - \eta)^2 + z_f^2},$$

$$R_\alpha = \sqrt{(\alpha - x_f)^2 + (y_f - \varphi)^2 + z_f^2},$$

and both waves are converging and have a geometric focus at the point

$$\vec{r}_f = \left\{x_f, y_f, z_f\right\} = \left\{f\cos\theta, \; f\sin\theta \; \sin\psi, \; f\sin\theta \; \cos\psi\right\};$$

c) the volume of integration in Eq. (1.2) is a small neighborhood of the geometric focus, i. e.,

$$\vec{r} = \vec{r}_f + \vec{\Delta}, \; (|\vec{\Delta}|/f) \sim (l/f) \ll 1; \tag{1.4}$$

d) for simplicity, the change in the polarization vectors is neglected, i.e., it is assumed that

$$e_{1y} \simeq \cos\psi, \quad e_{2z} \simeq -\sin\psi.$$

This description is entirely adequate for the real situation, when the focused laser beam is directed at angles θ and ψ into a crystal of length $2l$, cut at an angle of 90 to the longitudinal axis. The refraction on the entry face of the crystal only affects A_1 and A_2 in Eq. (1.3) — the functions are very "passive" in subsequent transformat ions and estimates. They can even be directly replaced by the characteristics of the external wave, if the surface of integration (in Kirchhoff's formula) is assumed to pass along the "external" surface of the crystal (see also the discussion of Eq. (1.6)).

Now, after substituting into Eq. (1.2) of the "linear" problem (E_{1y} and E_{2z} from Ref. 6) the expansions of the integrand in a series in $(|\Delta|/f)$ up to second-order terms and introducing t as the variable of integration, we obtain

$$\vec{H}_3(\vec{r}_0) = \vec{K}_1 \int\limits_{-1}^{+1} d\Delta_z \iint\limits_{-\infty}^{+\infty} d\Delta_x \, d\Delta_y \times \iint\limits_{-\infty}^{+\infty} d\xi \, d\eta \times$$

$$\times \iint\limits_{-\infty}^{+\infty} d\alpha \, d\varphi \; F_1 \exp(i\vec{Q}\vec{\Delta}). \tag{1.5}$$

We note that F_1 and Q do not depend on $\vec{\Delta}$, and the limits of integration over the transverse coordinates are infinite owing to the fact that the beam is spatially limited.

The first integral in Eq. (5) can be easily calculated and the next two integrals are equal to δ-functions, which removes the additional integration over two variables (for example, over ξ and η). After simple calculations we obtain

$$\vec{H}_3(\vec{r}) = \vec{K}_2 \iint\limits_{-\infty}^{-\infty} F_2(\alpha,\varphi) \; \mathrm{sinc}\left[lQ_z(\alpha,\varphi)\right] \; d\alpha d\varphi, \tag{1.6}$$

where $\mathrm{sinc}x = (\sin x)/x$.

It follows from Eq. (1.6) that sections of the boundary surface touching the curve

$$Q_z = Q_z(\vec{r}_0, \alpha, \varphi) = 0 \tag{1.7}$$

are responsible for the formation of the field H_3 at an arbitrary point r_0. We note that the form and position of the curve given by Eq. (1.7) do not depend on the specific form of the functions A_1 and A_2 from Eq. (1.3). The product of the latter, appearing in F_2, determines only the "energy weight", with which each point on the bounding surface (α, φ) enters into the integral (1.6).

After transferring to the coordinates q and p with the help of the relations

$$q = \frac{x_f - \alpha}{R_\alpha}, \; p = \frac{y_f - \varphi}{R_\alpha}, \; \alpha = x_f - \frac{z_f q}{\sqrt{1 - q^2 - p^2}},$$

$$\varphi = y_f - \frac{z_f p}{\sqrt{1 - q^2 - p^2}}$$

it is obvious that the curve (1.7) is an ellipse at the point (q_0, p_0):

$$q_0 = -\frac{d}{2}\frac{R'_0}{\beta^2 k_{3e} k_{20}}\frac{(x_f - x_0)}{T};$$

$$p_0 = -\frac{d}{2}\frac{R'_0}{k_{3e} k_{2e}}\frac{(y_f - y_0)}{T};$$

$$d = (k_{20}^2 - k_{10}^2) + \frac{k_{3e}^2 T}{R_0'^2};$$

$$T = \frac{(x_f - x_0)^2}{\beta^4} + (y_f - y_0)^2 + (z_f - z_0)^2;$$

$$R'_0 = \sqrt{\frac{(x_f - x_0)^2}{\beta^2} + (y_f - y_0)^2 + (z_f - z_0)^2},$$

and the ratio of the semiaxes is

$$\gamma = (z_f - z_0)/T.$$

The semiaxis of the ellipse makes an angle φ' with the q-axis of the coordinate system, and

$$\mathrm{tg}2\varphi' = \frac{2\beta^2(x_f - x_0)(y_f - y_0)}{\left[(x_f - x_0)^2 - \beta^4(y_f - y_0)^2\right]}.$$

We introduce another system of coordinates ρ and θ' tied to the

$$q(\rho, \theta') = q_0 + \frac{\rho\cos(\theta' - \varphi')}{\sqrt{\gamma^2\cos^2\theta' + \sin^2\theta'}},$$

$$p(\rho, \theta') = p_0 - \frac{\rho\sin(\theta' - \varphi')}{\sqrt{\gamma^2\cos^2\theta' + \sin^2\theta'}}, \tag{1.8}$$

where ρ is the minor semiaxis of the ellipse (1.7). For $\gamma = 1$ ($x_0 = x_f$, $y_0 = y_f$) corresponds to a polar coordinate system shifted linearly into the point (q_0, p_0) and rotated by an angle φ'.

In the coordinates ρ and θ' we obtain instead of (1.6)

$$\vec{H}_3(\vec{r}_0) = \vec{R}\int_0^{2\pi} d\theta' \int_0^1 F(\rho, \theta')\,\mathrm{sinc}[lU(\rho,\theta')]\rho\, d\rho, \tag{1.9}$$

where

$$B_1 = k_{3e}\frac{x_f - x_0}{\beta^2 R_0} + k_{20}q,$$

$$B_2 = k_{3e}\frac{y_f - y_0}{R'_0} + k_{20}p,$$

$$s = x_f + \frac{B_1 z_f}{\sqrt{k_{10}^2 - B_1^2 - B_2^2}},$$

$$t = y_f + \frac{B_2 z_f}{\sqrt{k_{10}^2 - B_1^2 - B_2^2}},$$

$$\vec{R} = 2\left[\vec{j}(z_f - z_0) - \vec{k}(y_f - y_0)\frac{n_{30}k_{10}k_{20}k_{3e}^2}{n_{3e}^2 R_0'^2} \times \right.$$
$$\left. \times\, l z_f^2 \chi_{123}\left[\omega_3\right]\sin 2\Psi \exp\left[ik_{3e}R'_0\right]\right],$$

$$U = k_{3e}\frac{z_f - z_0}{R'_0} + k_{20}\sqrt{1 - q^2 - p^2} +$$
$$+ \sqrt{k_{10}^2 - B_1^2 - B_2^2}.$$

The expression (1.9) determines the exatraordinary spherical wave at the sum frequency

$$\vec{H}_3(\vec{r}_0) = \left[\vec{j}h_y + \vec{k}h_z\right] A_3(\vec{r}_0)\, e^{ik_{3e}R'_0},$$

emanating from the geometric focus.

2. RESULTS

As the simplest example we shall study the case of Gaussian beams. Assume that in the coordinate system x', y', tied to the surface of the lens, the amplitudes A_1 and A_2 are equal and have the form

$$A_{1,2} = A_0\exp\left[-\frac{x'^2 + y'^2}{D^2}\right], \quad A_0 = \sqrt{\frac{16P}{cnD^2}}. \tag{2.1}$$

Stictly speaking, in Eq. (2.1) P is the average power of the radiation, since all fields are assumed to be monochromatic. However for rectangular pulses that are not too narrow, according to the quasistatic approximation,[3] the pulse power can be substituted instead of the average power without making an appreciable error. In addition, estimates show that to within ~ 1–5% it may be assumed that the integrand in Eq. (1.9) does not depend on θ'. Then

$$J = \int_0^{2\pi} d\theta' \int_0^1 F\,\mathrm{sinc}[lU]\,\rho d\rho \simeq 2\pi\int_0^1 F\,\mathrm{sinc}[lU]\,\rho d\rho. \tag{2.2}$$

The typical dependence of the amplitude on A_3 the coordinate x_0 of the observation point (y_0 = const), calculated on a computer using the formula (1.9) taking into account Eqs. (2.1) and (2.2), is presented in Fig. 1 for different focal lengths of the lens. The calculations were performed for wavelengths of a copper-vapor laser (CVL) λ_1 = 510.55 nm and λ_2 = 578.21 nm.

If the efficiency (η) of the process of generation of the sum frequency (GSF) is defined as $P_3/(P_1 + P_2)$, where P_1 is the pulse power in each line, then for $P_1 = P_2 = P$ we obtain from Eqs (1.9), (2.1), and (2.2)

$$\eta = \frac{bV^2Pc}{\pi^3 n_{3e} D^4}\left[z_f^2 l \sin 2\Psi\right]^2 \times$$

$$\times \iint\limits_{-\infty}^{+\infty}\left[J\,\frac{\sqrt{(y_f-y_0)^2+(z_f-z_0)^2}}{R_0'^{\,2}}\right]^2 dx_0 dy_0, \qquad (2.3)$$

$$V = 8\pi\,\frac{n_{30}k_{20}k_{3e}^2\chi_{123}(\omega_3)}{c n_{3e}^2 k_{10}\sqrt{n_{10}n_{20}}}.$$

$b = 1$ for the case of generation of the second harmonic (SHG), $b = 2$ for GSF, and c is the velocity of light.

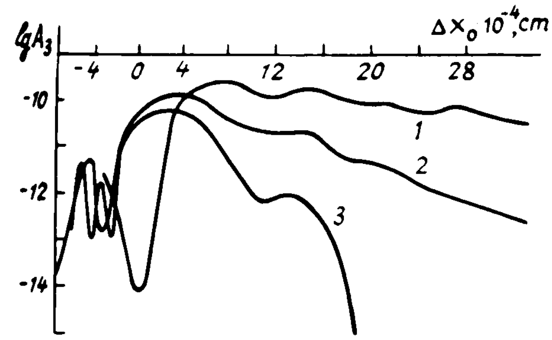

FIG. 1. The logarithm of the amplitude of the field at the sum frequency versus the coordinate x_0 of the observation point for different focal lengths of the lens f = 100 (1), 200 (2), *and* 300 cm (3).

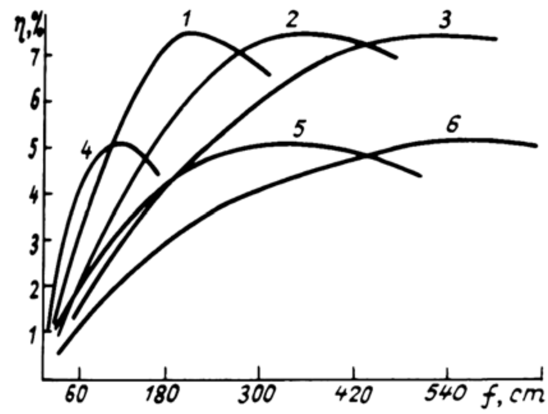

FIG. 2. The conversion efficiency versus the focal length of the lens for different radii D of the beam at the lens. GSF: 0.6 (1), 1.0 (2), 1.4 cm (3); *SHG*: 0.2 (4), 0.6 (5), and 1.0 cm (6).

Figure 2 shows the dependence of η on the focal length of the lens for different values of the beam radius at the lens for SHG of the yellow line and GSF of the CVL. The calculations were performed for pulsed power P = 5 kW (the corresponding average power is equal to 1 W). In both cases η reaches its maximum value at $\psi = 45$. The optical values of в are equal to 1.117 rad and 1.265 rad for SHG and GSF, respectively. Here it is evidently pointless to make a detailed analysis of the expression (2.3), since the results pertain primarily to the well-known results of the parabolic theory for Gaussian beams.

The situation represented by the relations presented above is obviously, to a certain extent. Ideal, and therefore Eq. (1.9) or (2.3) can be used correctly only for estimating the limiting, theoretically achieved values of the amplitude of the nonlinear field and conversion efficiency. The questions regarding refinement of the working model for specific experimental conditions are a separate subject and a complicated problem, and are not studied here.

3. SECOND HARMONIC GENERATION IN A FOCUSED BEAM. NONLINEAR GENERATION REGIME

In this section for simplicity we shall study only the second harmonic generation — the particular case (1.8).

We shall determine the contribution of an elementary section of the boundary surface $d\alpha\ d\varphi = \rho d\rho d\theta'$ to the amplitude of the field of the second harmonic. Introducing

$$L = \frac{2l}{\sin\theta\cos\Psi},\quad \sigma_2 = \frac{\pi k_{2e}}{n_{2e}^2}\chi_{123}(\omega_3)\sin\theta\sin 2\Psi,$$

$$\Delta_k = \left[\frac{k_2^e}{k_{2e}}\right]^2 \sin\theta\cos\Psi\, U(\theta',\rho),$$

$$\left[\frac{k_2^e}{k_{2e}}\right]^2\sin^2\theta + \left[\frac{k_2^e}{k_{20}}\right]^2\cos^2\theta = 1,$$

$$A_1^2(0) = \sqrt{(z_f-z_0)^2+(y_f-y_0)^2}\,\frac{n_{20}k_{20}^2k_{2e}^2}{\pi n_{2e}R_0'^{\,2}} \times$$

$$\times \frac{z_f\cos\Psi\, A_1(\alpha,\varphi)\, A_1(s,t)}{(k_{10}^2-B_1^2-B_2^2)(1-q^2-p^2)}, \qquad (3.1)$$

we obtain from Eq. (1.8)

$$\frac{|\Delta\vec{H}_2(\vec{r}_0)|}{\rho\, d\rho\, d\theta'} = n_{2e}L\sigma_2\,\mathrm{sinc}\left[\left[\frac{k_{2e}}{k_2^e}\right]^2\frac{L}{2}\Delta_k\right]. \qquad (3.2)$$

It is easy to see[3,6] that the right side of Eq. (3.2) corresponds exactly to the amplitude of the second-harmonic plane wave in the fixed-field approximation. Using Eq. (3.2) the following formal physical interpretation can be proposed for Eq. (1.9):

a) every point of the boundary surface is a source of a partial ray (plane wave) with amplitude $A_1(0)$ and direction determined by Δ_k;

b) every partial ray is converted into the second harmonic in accordance with the laws of the fixed-field approximation: and,

c) at each point $\vec{r}_0$ the amplitude of the total field of the second harmonic is determined by the superposition of this completely determined (for a selected point $\vec{r}_0$) infinite collection of independent "elementary contributions".

The geometric-optics (GO) approximation[3] makes it possible to represent an arbitrary complex field at the fundamental frequency as a collection of independent partial plane waves and thereby reduce the problem to finding the second harmonic wave for each such ray followed by integration over all possible directions. In our case we have the inverse problem. The structure of the plane waves at the fundamental frequency, corresponding to the solution, can be "constructed" from the known (1.8) exact solution for the second harmonic field. In contradistinction to geometric optics, each observation point $\vec{r}_0$ corresponds not to one ray of the fundamental wave but rather to an infinite collection of such independent partial rays.

These circumstances suggest, by analogy to geometric optics, that the amplitude of the field at the entry into the medium does not affect the structure and independence of the partial waves, and determines only the character of their transfer into the second harmonic. Then substituting in Eq. (3.2) the "fixed-field approximation" by the well-known solution[3] for plane waves In the strongly nonlinear regime, we obtain finally

$$\Delta\vec{H}_2(\vec{r}_0) = \frac{\vec{j}(z_f - z_0) - \vec{k}(y_f - y_0)}{\sqrt{(z_f - z_0)^2 + (y_f - y_0)^2}} \exp\left[ik_{2e}R'_0\right] \times$$

$$\times \int_0^{2\pi} d\Theta' \int_0^1 \sqrt{\frac{\sigma_2}{\sigma_1}}\, A_1(0)\, \sqrt{\kappa}\, \mathrm{sn}[W, \kappa]\rho\, d\rho, \qquad (3.3)$$

where $\sigma_1 = \dfrac{2\pi k_{10}}{n_{10}^2}\chi_{23}(\omega_1)\sin\Theta\sin 2\Psi$.

$$\sqrt{\kappa} = \sqrt{1 + \left[\frac{\Delta_1}{2}\right]^2} - \frac{\Delta_1}{2},$$

$$\Delta_1 = \left[\frac{k_{2e}}{k_2^e}\right]^2 \frac{\Delta_k}{2\sqrt{\sigma_1\sigma_2}\, A_1(0)},$$

$$W = L\sqrt{\sigma_1\sigma_2}\, A_1(0)/\sqrt{\kappa},$$

where sn $[W, \kappa]$ is Jacobi's elliptic sine.

The expression (3.3), being a generalization of the exact solution (1.9), incorporates all effects accompanying the process of focusing of the beam, and in addition it reflects a number of new features, characteristic for the nonlinear generation regime. In particular, as the amplitude of the fundamental wave increases the values of the optimal focal lengths,, which in the fixed-field approximation are found from the plots in Fig. 2, will change. In connection with the reverse transfer of energy from the harmonic to the fundamental wave, the monotonic character of the dependence of the efficiency on the length of the crystal, observed in the fixed-field approximation, will be destroyed, etc. It is obvious that for small values of the amplitude $A_1(0)$, $\Delta_1 \gg 1$ solution (3.3) goes over into (1.8).

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