EFFICIENCY OF A MULTIDITHER ADAPTIVE OPTICS SYSTEM WITH A MULTICHANNEL PHASE MODULATION

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Received June 5, 1995

Using the moment representation for noise in the control channels we have obtained an analytical representation for Strehl ratio. Using flexible and segmented mirrors as correctors the influence of correlated and uncorrelated Gaussian noise as well as of Poisson noise on the quality of an adaptive optics system is analyzed. It is shown that the Strehl ratio does not depend on the moment α_1 of the noise in the control channels.

1. INTRODUCTION

In optical sounding and communication through the turbulent atmosphere, multidither adaptive optics systems are used to correct wave front. Normally the aperture of an adaptive optics system is divided into m subapertures, each of them performs spatial phase modulation of the incident wave. Usually dithering of subapertures is performed simultaneously. Normally the following expansion of the functional of quality into a Taylor series[1] is used as a control signal in a system with a multichannel phase modulation:

$$I = A^2 \left[N + J_0^2(a_0) \sum_{i=1}^{m} \cos(\beta_i - \beta_j) \right] -$$
$$- 4 A^2 J_0(a_0) J_1(a_0) \sum_{i=1}^{m} \sum_{j=1}^{m} \sin(\omega_i t) \sin(\beta_i - \beta_j) +$$
$$+ 4 A^2 J_0(a_0) J_2(a_0) \sum_{i=1}^{m} \cos(2\omega_i t) \cos(\beta_i - \beta_j) + \ldots , \quad (1)$$

where I is the intensity at a point photodetector, $J_{0,1,2}$ are Bessel functions, a_0 and ω_i are the amplitude and the frequency of dithering, β_j is the phase at the jth subaperture, m is the number of control channels in the adaptive optics system.

To perform the control in actual systems the signal proportional to the second term of Eq. (1) extracted with a set of band-pass filters is normally used.[2] It is obvious that, along with the signal **X**, the noise is also present at the filter output. So the following vector of control signals **Y** is applied to the adaptive mirror

$$\mathbf{Y} = \mathbf{X} + \mathbf{n}, \quad (2)$$

where **n** is the vector of an additive noise with known moment characteristics, **X** is the vector of control signals calculated by the equation

$$\mathbf{X} = \mathrm{grad}\, I(\mathbf{B}) = \frac{\partial I}{\partial \beta_i}; \quad i = \overline{1, m};$$
$$\mathbf{B} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_m\}. \quad (3)$$

The element of an adaptive optics system, which directly governs the process of compensation for nonstationary phase aberrations, is the controlled mirror. The mirrors that have been developed up to now can be divided[3] into the flexible and segmented ones. Membrane mirrors, which have the response functions, localized in the vicinity of an actuator are distinguished among flexible mirrors. Flexible mirrors with the response functions close to the system of Zernike polynomials are developed on the basis of piezoelectric plates. Notwithstanding the variety of mirrors, in the case of a fast response when dynamic errors can be ignored, the problem of optimization is reduced to minimization of the rms error variance σ_a^2 of the wave front approximation and variance of the noise error σ_n^2

$$\min(\sigma_a^2 + \sigma_n^2). \quad (4)$$

In this problem the most difficult is the assessment of noise error influence on the efficiency of the adaptive system operation with noise of different origin in control channels. In this case it is reasonable to use Strehl ratio as the efficiency criterion. But as was pointed out in Ref. 4 Strehl ratio can be used only when the rms error is less than $\lambda/8 - \lambda/16$, that corresponds to the inequality

$$\sqrt{\sigma_a^2 + \sigma_n^2} < \lambda/8, \quad (5)$$

where λ is the wavelength of optical radiation.

At present the problem on calculation of Strehl ratio has been solved only for the case when **n** is a Gaussian noise[2]. In the adaptive optics system vector **Y** is formed based on the intensity distribution analysis. Therefore the noise statistics obeys the Poisson law.[5] Development of a corresponding algorithm allows one

0235-6880/96/03 209-05 $02.00

to assess the influence of this noise on the efficiency of adaptive optics system.

Thus, at present the derivation of analytical equation for the Strehl ratio of a multidither adaptive optics system is an important problem.

2. DERIVATION OF THE BASIC RELATIONSHIPS

Let us consider the following formulation of the problem. The adaptive optics system with a multichannel phase modulation[6] focuses radiation on a point reflector (Fig. 1). Signals proportional to the intensity gradient (Eq. (3)) are detected at the output of band-pass filters. Control signals are transferred to the input of the subsystem of generation of control forces. The control forces are applied to the input of the adaptive mirror. Let us assume that statistical characteristics of noise α_1, α_{11}, α_2 at the output of filters are known *a priori*. Calculating parameters of the adaptive optics system we can assess these characteristics by the well-known methods.[5]

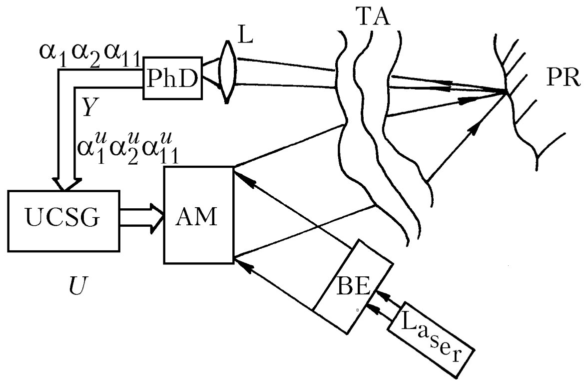

FIG. 1. Adaptive optics system with a multichannel phase modulation: L is a lens, TA is the turbulent atmosphere, PR is a point reflector, BE is a beam expander, AM is an adaptive mirror, UCSG is a unit of control signal generation, PhD is a photodetector.

In Refs. 1 and 2 the multidither algorithm is considered in which the set of band-pass filters detects the signal

$$x_{ij} = \sum_{j=1}^{m} \sin(\beta_i - \beta_j), \tag{6}$$

where x_{ij} is the control signal at the jth channel at the ith step.

The multidither algorithm allowing for the second term of expansion (1), according to Eq. (6) faces certain difficulties associated with the nonlinear dependence of the criterion of a system efficiency on the control coordinates and with the dependence of a control signal in each control channel on the control signals in other channels. In Refs. 6, 7, and 8 it was shown that in this case the most efficient is the following algorithm:

$$\mathbf{U}_{i+1} = \mathbf{U}_i + \mathbf{C}^{-1}\,\mathbf{Y}_i\,, \tag{7}$$

where $\mathbf{U}_i$ is the phase vector at the ith step of the adaptive control, $\mathbf{C}$ is the $m\times m$ matrix of coefficients. The matrix elements $c_{nk} = m$ when $n = k$ and $c_{nk} = -1$ when $n \neq k$. $\mathbf{C}^{-1}$ is the matrix inverse to the matrix $\mathbf{C}$. The elements of this matrix $c'_{nk} = 2/(m + 1)$ when $n = k$ and $c'_{nk} = 1/(m + 1)$ when $n \neq k$.

Let us consider the following equation that allows for the noise in the control channels of the adaptive optics system:

$$\mathbf{U}_{i+1} = \mathbf{U}_i + \mathbf{C}^{-1}\,(\mathbf{X}_i + \mathbf{n}). \tag{8}$$

The moments α_1, α_{11}, α_2 of the random values $\mathbf{n}$ are known *a priori*. Taking into account the linearity of the algorithm and additive character of noise let us find the corresponding moments α_1^u, α_{11}^u, α_2^u of $\mathbf{U}$ at $\mathbf{X} = 0$. From here on a superscript of a moment denotes random quantity and a subscript denotes the order of the moment. Thus the first-order moment is

$$\alpha_1^u = \langle \frac{1}{m+1}\,(2n_1 + n_2 + n_3 + \ldots + n_m)\rangle = \alpha_1. \tag{9}$$

From here on angular brackets denote averaging over an ensemble. Similarly we can find the moments

$$\alpha_2^u = \langle \frac{1}{m+1}\,(2n_1 + n_2 + n_3 + \ldots + n_m)^2\rangle =$$

$$= \frac{(m+3)\,\alpha_2 + (m^2 + m + 2)\,\alpha_{11}}{(m+1)^2}\,; \tag{10}$$

$$\alpha_{11}^u = \langle \frac{1}{(m+1)^2}\,(2n_1 + n_2 + n_3 + \ldots + n_m) \times$$

$$\times\,(n_1 + 2n_2 + n_3 + \ldots + n_m)\rangle =$$

$$= \frac{(m+2)\,\alpha_2 + (m^2 - 2m + 3)\,\alpha_{11}}{(m+1)^2}\,. \tag{11}$$

Once α_1^u and α_2^u are known, it is easy to find, in the multidither system, the noise variance at the output of the unit of the control signal generation:

$$D_{\text{out}} = \alpha_2^u - (\alpha_1^u)^2 =$$

$$= \frac{(m+3)\,\alpha_2 + (m^2 + m + 2)\,\alpha_{11}\,(m+1)^2\,\alpha_1}{(m+1)^2}\,. \tag{12}$$

Let us consider Eq. (11) in more detail. It is evident that the variance D_{out} is minimum when α_{11}^u, $\alpha_1^u \to 0$, i.e., when uncorrelated noise with zero mathematical expectation is present in the control channels. Obviously, thermal Gaussian noise well satisfies these conditions. Indeed, if $m \gg 1$

$$\lim_{\alpha_{11},\alpha_1 \to 0} D_{\text{out}} =$$

$$= \frac{(m+3)\alpha_2 + (m^2 + m + 2)\alpha_{11} + (m+1)^2\alpha_1}{(m+1)^2} \approx \frac{\alpha_2}{(m+1)}\,. \tag{13}$$

In the case when only shot uncorrelated Poisson noise is present in the control channels, the variance D_{out} significantly increases

$$\lim_{\alpha_{11}\to 0} D_{\text{out}} = \\ = \frac{(m+3)\,\alpha_2 + (m^2+m+2)\,\alpha_{11} + (m+1)^2\,\alpha_1}{(m+1)^2} \approx \alpha_1. \quad (14)$$

The correlation coefficient of the output signal α_{11}^u is nonzero even if the input noises **n** are uncorrelated.

$$\lim_{\alpha_{11}\to 0} \alpha_{11}^u = \frac{(m+2)\,\alpha_2 + (m^2-2m+3)\,\alpha_{11}}{(m+1)^2} \approx \frac{\alpha_2}{(m+1)}. \quad (15)$$

The main conclusion that can be drawn from the analysis of Eq. (12) is that even in the case of uncorrelated noise at the input of the control signal generation unit, noise at the input of adaptive mirror is always correlated.

Let us derive an equation for the Strehl ratio of an adaptive optics system. Assume that the plane wave front is incident on the surface of an adaptive mirror. Due to focusing in vacuum the spherical wave is formed in the plane of an adaptive mirror. This wave converges at the focal spot. According to the Huygens-Fresnel law the complex amplitude in the focus is

$$A = \int_S A_0 \, \mathrm{d}^2 r, \quad (16)$$

where A_0 is the complex amplitude of the wave emitted, which is assumed to be constant within the aperture of the optics system, s is the area of the adaptive mirror aperture.

If the emitted wave front is distorted due to the influence of noise vector **n** on the control signals, its deviation from the ideal wave front is $\varphi(r)$. Actually, the amplitude at the focus is

$$A' = \int_S A_0 \exp(\varphi(r)) \mathrm{d}^2 r. \quad (17)$$

Because the Strehl ratio is the ratio of intensity at the focal spot of a real system to the intensity in a system without aberrations it can be written as

$$\mathrm{St} = \frac{I}{I_0} = \frac{\langle |A' A'^*| \rangle}{|A_0 A_0^*|}, \quad (18)$$

where I_0 is the radiation intensity at the lens focus of an ideal system, I is the intensity at the lens focus in the presence of noise.

Equation (18) can be re-written in the following form:

$$\mathrm{St} = \frac{1}{s^2} \iint_s \langle \exp\{i\,\varphi(r_1) - i\,\varphi(r_2)\}\rangle \, \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2 =$$

$$= \frac{1}{s^2} \iint_s \left\langle \frac{|A[\cos(\varphi(r_1)) + i\sin(\varphi(r_1))]\; A[\cos(\varphi(r_2)) - i\sin(\varphi(r_2))]|}{|A\,A|} \right\rangle \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2 =$$

$$= \frac{1}{s^2} \iint_s \left\langle \sqrt{[\cos^2(\varphi(r_1) - \varphi(r_2)) - \sin^2(\varphi(r_1) - \varphi(r_2))]} \right\rangle \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2. \quad (19)$$

Because the terms $\varphi(r_1)$, $\varphi(r_2)$ are small, the second summand in Eq. (19) can be neglected. Expanding Eq. (19) into a Taylor series and keeping only the first two terms we obtain

$$\mathrm{St} = 1 - \frac{1}{s^2} \iint_s \frac{\langle (\varphi(r_1) - \varphi(r_2))^2 \rangle}{2} \, \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2. \quad (20)$$

For further analysis let us introduce into the P-space a set of orthonormal functions **Z** with a scalar product

$$\{z_i(r)\; z_j(r)\} = \int_s z_i(r)\; z_j(r)\; \mathrm{d}^2 r$$

and with the following condition of orthonormality:

$$\{z_i(r)\; z_j(r)\} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \quad (21)$$

To describe the surface of an adaptive mirror we use the system of response functions $S_i(r)$ from the P-space

$$\varphi(r) = \sum_{i=1}^{m} S_i(r)\; x_i, \quad (22)$$

where $\varphi(r)$ is the mirror response to the vector of control signals **X.** By substitution of Eq. (22) into Eq. (20) we obtain

$$\mathrm{St} = 1 - \frac{1}{2s^2} \iint_s \left\langle \left(\sum_{i=1}^{m} S_i(r_1)\, n_i - \sum_{j=1}^{m} S_j(r_2)\, n_j \right)^2 \right\rangle \mathrm{d}^2 r_1 \mathrm{d}^2 r_2. \quad (23)$$

3. SEGMENTED MIRROR

Let us consider Eq. (22) for a segmented mirror in more details. The system of response functions S_s of this mirror satisfies the condition (20). For this functions we obtain

$$S_{si}(r) = 1 \quad \text{when} \quad r \in \Omega_i, \tag{24}$$

where Ω_i is the area of the ith mirror subaperture, $\Omega_i = s/m$.

The Strehl ratio can be written in the following form:

$$\mathrm{St} = 1 - \frac{1}{2s^2} \iint\limits_s \left\langle \left[\sum_{i=1}^{m} S_{si}(r_1)\, n_i \right]^2 - 2 \left[\sum_{j=1}^{m} S_{sj}(r_2)\, n_j \sum_{i=1}^{m} S_{si}(r_1)\, n_i \right]^2 + \left[\sum_{j=1}^{m} S_{sj}(r_2)\, n_j \right]^2 \right\rangle \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2. \tag{25}$$

Considering Eq. (22) and assuming $s = \Omega_i m$ we obtain

$$\mathrm{St} = 1 - \frac{1}{2} \left[\frac{\alpha_2^u}{m\Omega} \sum_{i=1}^{m} \Omega_i - \frac{2\alpha_{11}^u}{m\Omega} \sum_{i=1}^{m} \Omega_i - \frac{\alpha_2^u}{m\Omega} \sum_{i=1}^{m} \Omega_i \right] = 1 - (\alpha_2^u - \alpha_{11}^u), \tag{26}$$

where α_2^u, α_{11}^u are moments of the random variable $\mathbf{U}$ defined with account for Eqs. (8), (9), and (10).

In accordance with the properties of moments[9]

$$\alpha_2^u - \alpha_{11}^u = (\kappa_2 + \kappa_1^2) - (\kappa_{11} + \kappa_1^2) = \kappa_2 - \kappa_{11}, \tag{27}$$

where κ_2, κ_1, κ_{11} are corresponding cumulants of the random variable $\mathbf{U}$.

Finally, considering the latter equation, we can write Eq. (26) in the following form

$$\mathrm{St} = 1 - (\kappa_2 - \kappa_{11}). \tag{28}$$

Substituting Eqs. (8), (9), and (10) into Eqs. (26) we obtain the following dependence of the Strehl ratio of an adaptive optics system on the statistic characteristics of noise at the output of band-pass filters as

$$\mathrm{St} = 1 - (\alpha_2^u - \alpha_{11}^u) = 1 - [\alpha_2 + (3m - 1)\,\alpha_{11}]/(m + 1)^2. \tag{29}$$

Equation (29) can be written in the form convenient for practical calculations

$$\mathrm{St} = 1 - (D_{\text{out}} + m^2 - k), \tag{30}$$

where D_{out}, m, and k are the variance, mathematical expectation, and a noise correlation coefficient at the input of the adaptive mirror;

$$D_{\text{out}} = \alpha_2^u - (\alpha_1^u)^2; \quad m = \alpha_1^u; \quad k = \alpha_{11}^u.$$

Obviously, assuming that m, $k = 0$ the well-known equation[2] for the Strehl ratio of uncorrelated Gaussian noise can be obtained

$$\mathrm{St} = 1 - D_{\text{out}}. \tag{31}$$

4. FLEXIBLE MEMBRANE MIRROR

The main difficulty in obtaining relations convenient for calculations is that the response functions of a flexible mirror do not satisfy the conditions of Eq. (21). Usually[10] such functions are represented in the following form:

$$S_{mi} = \exp(-\, a(r - r_i)^b), \tag{32}$$

where a and b are construction factors of a membrane mirror.

For further analysis let us introduce the matrix S of the membrane mirror response[8]:

$$1/c \int\limits_S S_{mi}\, S_{mj}\, \mathrm{d}^2 r = S_{ij}, \tag{33}$$

where c is the normalizing factor, chosen in the way to make the diagonal elements of the matrix equal to unity: $S_{ii} = 1$, $i = 1, m$.

The Strehl ratio can be written in the following form:

$$\mathrm{St} = 1 - \frac{1}{2s^2} \iint\limits_s \left\langle \left[\sum_{i=1}^{m} S_{mi}(r_1)\, n_i \right]^2 - 2 \left[\sum_{j=1}^{m} S_{mj}(r_2)\, n_j \sum_{i=1}^{m} S_{mi}(r_1)\, n_i \right]^2 + \left[\sum_{j=1}^{m} S_{mj}(r_2)\, n_j \right]^2 \right\rangle \mathrm{d}^2 r_1\, \mathrm{d}^2 r_2, \tag{34}$$

where S_{mj} is a response function of a flexible membrane mirror.

Allowing for Eq. (33), the Eq. (34) is written as

$$\mathrm{St} = 1 - \left[\alpha_2^u \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} - \alpha_{11}^u \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} \right]. \tag{35}$$

By introducing for matrix elements the operator of summation sum(*), Eq. (35) can be written in a compact matrix form

$$\mathrm{St} = 1 - \left[(D_{\text{out}} + m^2 - k)\, \frac{\mathrm{sum}(\mathbf{S})}{m} \right] = 1 - \left[\frac{\alpha_2 + (3m - 1)\,\alpha_{11}}{(m + 1)^2} \right] \frac{\mathrm{sum}(\mathbf{S})}{m}. \tag{36}$$

In this formula Eq. (29) has been taken into account. It is obvious that, if instead of **S** we substitute the unit matrix into Eq. (36) which is a response function of a segmented mirror, we obtain equation (30):

$$\lim_{\substack{s_{ij}\to 0\\ s_{ii}\to 1}} \left\{1 - \left[(D_{\text{out}} + m^2 - k)\,\frac{\text{sum}(\mathbf{S})}{m}\right]\right\} =$$
$$= 1 - (D_{\text{out}} + m^2 - k). \qquad (37)$$

Thus Eq. (37) is a generalization of Eq. (29) that describes a flexible membrane mirror. Analyzing Eq. (37) one can conclude that Strehl ratio of an adaptive flexible mirror depends significantly on the position of actuators and on the form of response functions. The obtained relation permits one to assess the efficiency of adaptive mirror with different types of actuator geometry at the preliminary stage of designing.

Using Eq. (29) it is possible to assess the influence of Gaussian both correlated and uncorrelated noise as well as Poisson noise on the value of Strehl ratio. It is an important advantage of this equation. Thus, for example, in the case of uncorrelated Poisson noise we obtain

$$\text{St} = 1 - (\lambda + \lambda^2)\ (\text{sum}(\mathbf{S})/m), \qquad (38)$$

where λ is the parameter of the Poisson distribution.

5. CONCLUSIONS

The analysis of Eqs. (29), (37), and (38) permits one to draw some important conclusions. The main factor that causes the decrease of Strehl ratio of a multidither adaptive optics system with a multichannel phase modulation is the correlation coefficient α_{11} of noise at the output of the system of band-pass filters. It seems so that, in order to decrease the influence of noise in the control channels on the performance of the adaptive optics system with a multichannel phase modulation, special attention should be paid to the frequency differentiation in the control channels.

It should be noted that equations (29), (37), and (38) allows one to assess the contribution of an individual moment into the value of the Strehl ratio. Also we have illustrated that Strehl ratio is independent of the moment α_1 which has the physical meaning of mathematical expectation of the noise vector. It likely can be explained considering that the Strehl ratio is independent of the phase variations averaged over the aperture of the adaptive optics system. Choosing the construction characteristics of a membrane adaptive mirror one should try to obtain its response matrix as close to the unit matrix as possible $\mathbf{I} \to \mathbf{S}$. In this case Strehl ratio is maximum.

It would be a mistake to conclude from the analysis of Eqs. (29), (37), and (38) that $\text{St} \to 1$ at $m \to \infty$, because α_2 and α_{11} are the functions of m and $\alpha_1 \sim m^4$ as it follows from Ref. 2. Choosing the number of control channels one should take this fact into account in the analysis of the whole adaptive optics system.

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