Light scattering by a finite-length cylinder in Wentzel–Kramers–Brillouin approximation. 3. Light scattering phase function

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Received September 27, 2007

Equations for calculating small-angle phase function (or element of scattering matrix f_{11}) and small-angle amplitude of light scattering by an optically "soft" finite-length circular cylinder have been derived in the Wentzel–Kramers–Brillouin (WKB) approximation for incident light, normal to the cylinder axis. The more precise general equation for light scattering amplitude by an optically "soft" finite-length circular cylinder is given in WKB approximation for incident light, normal to the cylinder axis.

The study of light scattering by non-spherical particles, which are components of natural and anthropogenic aerosols, hydrosol suspensions, and ice crystals, is of great importance for monitoring air and ocean, in colloid chemistry, etc.[1]

The Rayleigh–Gans–Debye (RGD), anomalous diffraction, and Wentzel–Kramers–Brillouin (WKB) approximations are convenient to be used when dealing with optically "soft" ($|m-1| \ll 1$, where m is the relative refractive index of the particulate matter) light scattering by non-spherical particles.

For optically "soft" particles with sizes larger than the wavelength, the most part of the scattered energy is usually concentrated in small scattering angles.[3] In this work, the small-angle scattering phase function of a finite-length circular cylinder is analyzed in WKB approximation for incident light, normal to the cylinder axis.

1. General equation for light scattering amplitude

To take into account spatial light scattering correctly, it is necessary to introduce two angles in two mutually perpendicular planes, i.e., the scattering angle β, measured from the forward scattering direction (along the y axis) and the additional azimuth scattering angle α, measured from the z axis (Fig. 1).

Using an integral expression for light scattering amplitude, obtain the following scalar equation in WKB approximation,[4,5] more complex in comparison with Ref. 4:

$$f(\alpha,\beta) = \frac{\rho^2 H\,(m-1)}{2\pi}\left[\frac{\sin\left(\frac{kH}{2}\cos\alpha\right)}{\frac{kH}{2}\cos\alpha}\right]\times$$
$$\times\int_0^1\int_0^{2\pi}\exp\left[i\left(\rho\psi_1(\alpha,\beta,t,\varphi)+\frac{\Delta}{2}\psi_2(t,\varphi)\right)\right]\mathrm{d}\varphi\, t\mathrm{d}t, \quad (1)$$

where $\psi_1(\alpha,\beta,t,\varphi) = t(\sin\alpha\sin\beta\cos\varphi + [m - \sin\alpha\cos\beta]\times\sin\varphi)$, $\psi_2(t,\varphi) = \sqrt{1-t^2\cos^2\varphi}$; a is the cylinder radius; H is the cylinder height; $\Delta = 2\rho(m-1)$ is the phase shift, $\rho = ka$ is the diffraction parameter of the cylinder; $k = 2\pi/\lambda$ is the wave number; λ is the wavelength in a dispersion medium.

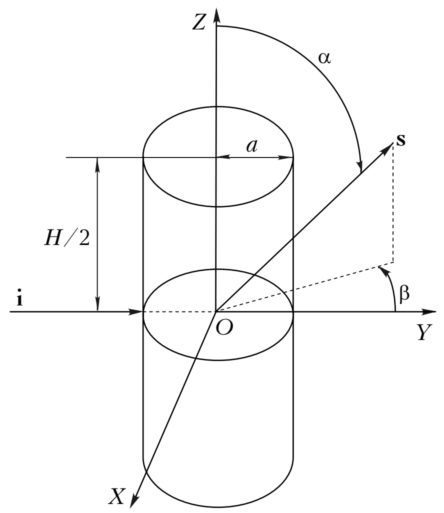

Fig. 1. Geometry of light scattering by a circular cylinder of a in radius and H in height (**i** and **s** are unit vectors of incident and scattered light, respectively).

However, if the azimuth scattering angle is close to the forward direction, i.e., $\alpha \to \pi/2$, then equation (1) takes the form, obtained by us in Ref. 4.

Again, restrict ourselves to the case of $\alpha = \pi/2$, which essentially simplifies the following equations, though somewhat deteriorates their generality.

2. Small-angle light scattering amplitude

At small scattering angles $\beta \ll 1$ and $\alpha = \pi/2$, $\sin\beta \approx \beta$, $\cos\beta \approx 1$, $\sin\alpha = 1$, and $\cos\alpha = 0$, i.e.,

$\rho\psi_1(\alpha, \beta, t, \varphi) = t[\rho\beta\cos\varphi + \Delta\sin\varphi/2]$; hence, from Eq. (1) obtain

$$f\left(\frac{\pi}{2},\beta\right) = \frac{\rho H}{2}\left\{\sum_{s=0}^{\infty}\frac{(-1)^s(\rho\beta)^{2s}}{2^s s!}\left[\frac{J_{s+1}(\Delta)}{\Delta^s} + i\frac{H_{s+1}(\Delta)}{\Delta^s}\right]\right\}, \quad (2)$$

where $J_s(x)$ and $H_s(x)$ are the Bessel function of the first kind and the s-order Struve function.

3. Small-angle scattering phase function

The light scattering phase function (or the scattering matrix element f_{11}) for natural light (chaotic polarization) is calculated as

$$f_{11}(\beta) = k^2|f(\beta)|^2\,\frac{1+\cos^2(\beta)}{2}, \quad (3)$$

where $|f(\beta)|^2$ is the squared modulus of light scattering amplitude.

Using Eqs. (2) and (3), we obtain the small-angle scattering phase function:

$$\frac{f_{11}(\beta)}{f_{11}(0)} = \sum_{j=0}^{\infty}\frac{(-1)^j c_j(\Delta)}{(2\Delta)^j\left[J_1^2(\Delta) + H_1^2(\Delta)\right]}(\rho\beta)^{2j}, \quad (4)$$

where

$$c_j(\Delta) = \begin{cases} \dfrac{J_{n+1}^2(\Delta) + H_{n+1}^2(\Delta)}{[(n)!]^2} + \\ +2\displaystyle\sum_{s=0}^{n-1}\frac{J_{s+1}(\Delta)J_{j-s+1}(\Delta) + H_{s+1}(\Delta)H_{j-s+1}(\Delta)}{s!(j-s)!} \\ \text{for even } j,\ j = 2n, \\ 2\displaystyle\sum_{s=0}^{n}\frac{J_{s+1}(\Delta)J_{j-s+1}(\Delta) + H_{s+1}(\Delta)H_{j-s+1}(\Delta)}{s!(j-s)!} \\ \text{for odd } j,\ j = 2n+1, \end{cases}$$

$n = 0, 1, 2, 3, \ldots$.

The series in Eq. (4) converges rapidly. For example, four terms of the series are required at $m = 1.03$ and $\rho\beta \le 1$ for obtaining an error less than 5%, while at $\rho\beta \le 2$ seven terms are required. However, large diffraction parameters $\rho > 5$–10 should be used for accurate calculation of small-angle phase function in Eq. (4).

The imaginary part of the light scattering amplitude (2) can be negligible as compared to the real one at small phase shifts $\Delta < 1$; hence, equation (4) will be somewhat simplifies:

$$\frac{f_{11}(\beta)}{f_{11}(0)} = 1 - \left[\frac{J_2(\Delta)}{\Delta J_1(\Delta)}\right](\rho\beta)^2 + \left[\frac{J_1(\Delta)J_3(\Delta) + J_2^2(\Delta)}{4\Delta^2 J_1^2(\Delta)}\right](\rho\beta)^4 + O\left((\rho\beta)^6\right). \quad (5)$$

Then, expanding the Bessel function in a series in Eq. (5) at small phase shifts $\Delta < 1$, obtain the equation

$$\frac{f_{11}(\beta)}{f_{11}(0)} = 1 - \left[\frac{1}{4}\right](\rho\beta)^2 + \left[\frac{5}{192}\right](\rho\beta)^4 + O\left((\rho\beta)^6\right) = \left[\frac{2J_1(\rho\beta)}{\rho\beta}\right]^2, \quad (6)$$

similar to the equation for phase function in RGD approximation at large diffraction parameters $\rho > 1$.[5,6]

On the contrary, at large phase shifts $\Delta > 10$, the real part in light scattering amplitude (2) can be totally negligible as compared to the imaginary one, which results in

$$\frac{f_{11}(\beta)}{f_{11}(0)} = 1 - \left[\frac{H_2(\Delta)}{\Delta H_1(\Delta)}\right](\rho\beta)^2 + \left[\frac{H_1(\Delta)H_3(\Delta) + H_2^2(\Delta)}{4\Delta^2 H_1^2(\Delta)}\right](\rho\beta)^4 + O\left((\rho\beta)^6\right). \quad (7)$$

Then, when Δ-series expanding the Struve function in Eq. (7) at $\Delta \to \infty$, obtain

$$\frac{f_{11}(\beta)}{f_{11}(0)} = 1 - \left[\frac{1}{3}\right](\rho\beta)^2 + \left[\frac{2}{45}\right](\rho\beta)^4 + O\left((\rho\beta)^6\right) = \left(\frac{\sin(\rho\beta)}{\rho\beta}\right)^2. \quad (8)$$

Note, that equation (8) coincides with the phase function for the Fraunhofer diffraction by a long slit of 2ρ in width.[7]

Figure 2 shows the dependence of the small-angle light scattering phase function $f_{11}(\beta)/f_{11}(0)$ on $\rho\beta$ for an infinitely long cylinder (rigorous solution) and a finite-length circular one in WKB approximation for the refractive index $m = 1.03$ and several phase shifts.

The algorithm from Ref. 8 was used in calculations for the infinitely long cylinder: Eq. (4) with seven terms in the series – for the finite-length circular cylinder in WKB approximation, Eqs. (6) and (8) – for RGD approximation and the Fraunhofer diffraction, respectively

At small phase shifts, small-angle WKB phase function (4) evidently is transforms into corresponding RGD phase function (6) (Fig. 2*a*), while it asymptotically tends to Fraunhofer diffraction equation (8) at large phase shifts (Fig. 2*b*). Note, that characteristic damped oscillations of small-angle phase function are observed with an increase of phase shifts similarly to the integral light scattering phase function of spherical particles.[9]

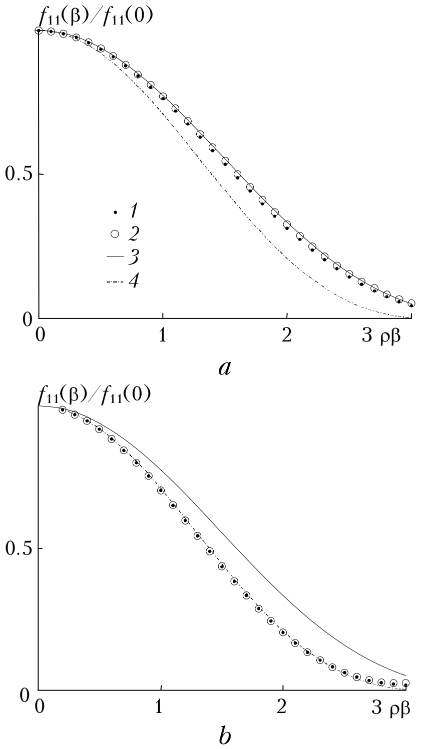

Fig. 2. Normalized small-angle light scattering phase function $f_{11}(\beta)/f_{11}(0)$ as a function of $\rho\beta$ for an infinitely long (*1*) and finite-length circular cylinders in WKB (*2*) and RGD (*3*) approximations and for the Fraunhofer diffraction (*4*) at the refraction index $m = 1.03$ and phase shifts $\Delta = 0.72$ (*a*) and $\Delta = 90$ (*b*).

Conclusion

A more precise general equation for light scattering amplitude by an optically "soft" finite-length circular cylinder is given in WKB approximation for incident light, normal to the cylinder axis, as well as the equations for small-angle phase function and light scattering amplitude at different phase shifts Δ. Asymptotic expressions for small-angle scattering phase function for the above cylinder are derived at small ($\Delta < 1$) and large ($\Delta \to \infty$) phase shifts.

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