SCINTILLATION SPECTRA DURING OBSERVATIONS OF STAR OBSCURATIONS BY THE EARTH'S ATMOSPHERE

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To calculate scintillation spectra one assumes that in the atmosphere the field of the fluctuations of the relative refractive index is locally isotropic on the sphere and homogeneous in height. In the phase-screen approximation scintillation spectra are expressed in terms of spectra of the refractive index fluctuations. The sphericity of the atmosphere is shown to affect the anisotropic fluctuations of the refractive index with an anisotropy coefficient of more than $(a_e/H_0)^{1/2}$. *Gaussian and power spectra with an arbitrary coefficient of anisotropy are also considered.*

Observations of star obscurations by planets[1,2] and of an interplanetary station radiosetting behind planets[3,4] definitely indicate that there exist inhomogeneities in the refractive index due to turbulence in Planetary atmospheres. Observations made from satellites[5,6] provide a possibility of investigating the atmosphere by means of obscuration data. The use of such methods from an extraterrestrial source during obscurations by the Earth's atmosphere are technically the simplest measurements that are possible when observing from onboard space platforms.

Relations connecting the fluctuation spectrum of the measured energy characteristics of a received signal with the spectrum of the refractive index have been described in a number of papers[7,8], in which statistical homogeneity of the refractive index fluctuations was assumed.

However, observations[6,9] definitely indicate that inhomogeneities of the refractive index are greatly extended over the Earth's surface, and in this case the assumption of their statistical homogeneity in cartesian coordinates is not sufficiently justified because of atmospheric sphericity. A more natural model be constructed based on the assumption that the relative refractive index fluctuations constitute a random field with properties close to those of fields which are statistically homogeneous on the sphere. Their properties were studied in Ref. 10. Using this model, the spectra of the phase fluctuations on an equivalent screen are calculated for the case of a plane wave from an extraterrestrial source and the observer located outside the atmosphere in the phase-screen approximation[11].

In this paper the fluctuation spectrum of the light flux from a star observed through the atmosphere from space platform is calculated for a preassigned refractive index spectrum, taking into account atmospheric refraction and sphericity. The values of the parameters which characterize the anisotropy are found on the basis of concrete spectra of the refractive index, the effect of the atmospheric sphericity on which is substantial.

The index of refraction of atmospheric air n differs only slightly from unity: $N = n - 1 \ll 1$. The marginal refractive index N can be represented as a random field, and we assume its statistical mean $\langle N \rangle$ to depend only on altitude h above the spherical (with radius a_e) Earth's surface in the region important for the solution of the scintillation problem. Without loss of generality, one can assume that in some height interval in the neighborhood of the fixed altitude h_p we have

$$\langle N \rangle = \bar{N}(h) = \bar{N}(h_p) \exp(-(h-h_p)/H_0(h_p)), \tag{1}$$

where H_0 is the altitude of the homogeneous atmosphere.

Relative fluctuations of the marginal refractive index $\nu = (N - \bar{N})/\bar{N}$ are small compared to unity, and, from the definition it follows from that $\langle \nu \rangle = 0$. Suppose that in the region important for scintillation calculations the structure function $D_\nu = \langle (\nu(\mathbf{R}_1 - \mathbf{R}_2))^2 \rangle$ depends on the difference of altitudes $\Delta h = h_1 - h_2$ and on the angular distance θ between vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ (see Fig. 1). In other words, the random field v is assumed to be locally homogeneous at any altitude and locally isotropic on the sphere[10]. The structure function $D_\nu(0, a_e\theta)$ can be estimated from measurements of the refractive index taken by an aircraft flying at a constant altitude, and $D_\nu(\Delta h, 0)$ may be estimated by means of freely ascending and descending sondes. Later it will be more convenient to use spectra $\Phi_\nu(\kappa_1, \kappa)$ in terms of which D_ν is given as follows:

$$D_\nu = 4\pi \int \kappa \, d\kappa \, d\kappa_1 \, \Phi_\nu(\kappa_1, \kappa) \left[1 - J_0(\kappa a_1 \theta) \cos(\kappa_1 \Delta h)\right], \tag{2}$$

where $\kappa^2 = \kappa_2^2 + \kappa_3^2$ and $J_0(\xi)$ is the Bessel function of the first kind. Equation (2) is valid under the assumption that the correlation in the random field on the sphere rapidly approaches zero for $a_e\theta \geq 1$.

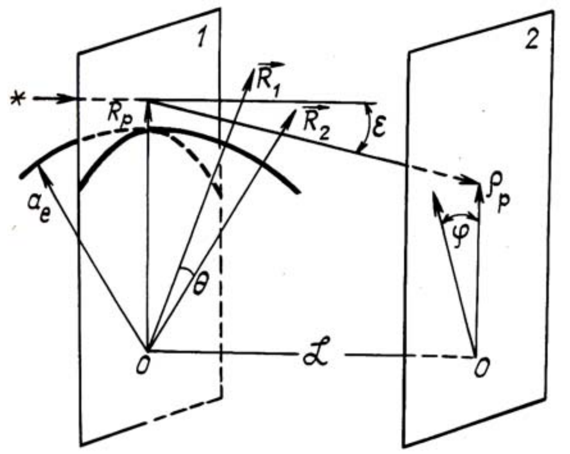

Fig. 1. Schematic representation of observation. 1 — the plane of the equivalent phase screen; 2 — the selected plane of observation.

We shall solve the problem of the flicker spectra when observing an extraterrestrial source from space platforms through the Earth's atmosphere by modeling the effect of the atmosphere by an equivalent screen. Such a possibility is based on the fact that fluctuations of the refractive index significantly affect light waves only in a limited layer of the atmosphere within the beam perigee due to the exponentially decreasing density. The thickness of this layer is H_0 and, consequently, the influence of atmospheric turbulence is concentrated in a segment of the beam with length of the order of $2(2a_eH_0)^{1/2}$. If at the exit of this layer the intensity fluctuations are not strong, and the distance L from an observer on board the space platform to the beam perigee is long enough, $(L \gg 1/2(2a_eH_0)^{1/2})$, then it is the case that the observed scintillations result from the phase modulation of that light waves by the atmospheric turbulence. We shall assume the equivalent phase screen to be in a plane perpendicular to the light rays incident upon the atmosphere which cuts through the Earth's center (see Fig. 1). After passing through the screen, a plane wave changes its phase to $\kappa\Phi$ where k is the wave number and

$$\Psi = \int N \, dl \tag{3}$$

is the perturbation of the eikonal due to the atmosphere. The integral in Eq. (3) is to be calculated over an interval along the beam. The radius of curvature of the beam in the atmosphere is quite large compared with the Earth's radius. Thus, within the beam perigee, where the curvature is at its maximum, the radius of curvature, which is the same order of magnitude as $H_0/\bar{N}$, exceeds a_e at altitudes greater than 20 km (these altitudes will be of interest to us later) and tens to hundreds of times greater than a_e. Therefore, when calculating Eq. (3) in the framework of our star scintillation problem, one can perform the integration over the straight line which is the continuation of the incident beam. This beam is determined by the sighting parameter R_p and by an angle φ whose vertex is at the center of the Earth and which lies in the phase screen plane.

The difference between $h_p = R_p - a_e$ and the altitude of the actual beam perigee is $a_e\bar{N}$ and is not large compared with h since $\bar{N} < 3{\times}10^{-5}$ at altitudes above 20 km. In what follows we will assume that $\Psi = \Psi(R_p,\ a_e\varphi_1)$. In the model used for the refractive index distribution the mean value $\langle\Psi\rangle$ depends only on R_p and is equal (with the accuracy up to H_0/a_e) to

$$\langle\Psi\rangle = \bar{\Psi}(R_p) = (2\pi a_e H_0)^{1/2}\, \bar{N}(h_p). \tag{4}$$

For this model of turbulence the structure function

$$D_\psi = \langle(\Psi(R_{p,1}, a_e\varphi_1) - \Psi(R_{p,2}, a_e\varphi_2))^2\rangle \tag{5}$$

of the eikonal fluctuations $\psi = \Psi - \bar{\Psi}$ in the plane behind the phase screen may be approximately represented by a spectral expansion with spectrum F_ψ. Using the results of Ref. 11 and assuming that $\bar{N}(h)$ is a smoothly varying function as described in Ref. 12, one can represent the spectrum $F_\psi(\kappa_1,\ \kappa_2)$ as follows:

$$F_\psi(\kappa_1,\kappa_2) = 2\pi\ a_1\ H_0\ \frac{(\bar{N}((h_1+h_2)/2)))^2}{(1+\kappa_1^2H_0^2)^{1/2}} \times$$

$$\times \int_{-\infty}^{\infty} \Phi(\kappa_1,\kappa) \exp\left(\frac{a_1 H_0 \kappa_3^2}{1+\kappa_1^2 H_0^2}\right) d\kappa_3. \tag{6}$$

The regular dependence of $\langle\Psi\rangle$ on the sighting parameter causes beams passing through the phase screen to deflect from their initial path by a refraction angle $\varepsilon = d\bar{\Psi} / dR_p$ in the plane containing the incident beam and the Earth's center. Since ε in turn depends on R_p, the average intensity of light $\langle I\rangle$ reaching the observer (the observer being located in a plane parallel to the phase screen at a distance L where $L \ll a_e/|\varepsilon|$ differs from the intensity of light incident upon the screen $I_3 = 1$, by a factor of q, where

$$q = (1 + L\, d^2\bar{\Psi} / d^2R_p)^{-1}, \tag{7}$$

The value R_p and the distance between the observation point and the origin of coordinates in the observation plane p are related as follows

$$\rho_p = R_p + \varepsilon L,\ \ \varepsilon<0.$$

Neither the field of eikonal fluctuations nor the field of intensity fluctuations in the observation plane is statistically homogeneous. However, suppose on the basis of the above-mentioned turbulence model that the correlation of the intensity to be vanishes. Then if the observation points are spaced at a distance greater than H_0 or if the angular distance between them is large $|\varphi_1 - \varphi_2| > 1$, then the correlation of the relative fluctuations of the intensity

$$B_I = q_1^{-1} q_2^{-1} \langle (I(\rho_1, a_e\varphi_1) - q_1)\ (I(\rho_2, a_e\varphi_2) - q_2) \rangle \tag{8}$$

can be approximately written in terms of the spectrum $F_I(\kappa_1\kappa_2)$

$$B_I = \int d\kappa_1 \exp\left(i\kappa_1(\rho_1 - \rho_2) + i\kappa_2 a_e(\varphi_1 - \varphi_2)\right) F_I(\kappa_1, \kappa_2). \tag{9}$$

The spectrum F_I is associated with the spectrum of the eikonal by the relation[7,8]

$$F_I(\kappa_1, \kappa_2) = 4k^2 q^{-1} \sin^2\left[\frac{L}{2k} \left(\frac{\kappa_1^2}{q} + \kappa_2^2 \right) \right] F_\psi(\kappa_1/q, \kappa_2),$$

$$q^2 = q_1 q_2. \tag{10}$$

Equations (6) and (10) give us a general solution of the problem of the scintillation spectrum when a star is observed through the Earth's atmosphere in the case of weak scintillations when the variance σ_I^2 is small

$$\sigma_I^2 = \int F_I(\kappa_1\kappa_2)\, d\kappa_1\, d\kappa_2 \ll 1. \tag{11}$$

For the anisotropic spectrum

$$\Phi_\nu = \frac{\sigma_\nu^2}{(2\pi)^{3/2}\kappa_m^2 \kappa_M} \exp\left[-\frac{\kappa_1^2}{2\kappa_M^2} - \frac{\kappa_2^2 + \kappa_3^2}{2\kappa_m^2} \right], \tag{12}$$

where σ_ν^2 is the variance, and $2\pi\kappa_M^{-1}$ and $2\pi\kappa_m^{-1}$ are the characteristic scales for the field of the refractive index fluctuations, it is not difficult to calculate F_I and in this instance to elucidate the conditions under which the sphericity of the atmosphere must be taken into account. Substituting Eq. (12) into Eq. (6), after some simple calculations we obtain

$$F_I(\kappa_1\kappa_2) = \frac{2k^2\sigma_\nu^2\, \bar{\Psi}^2 \exp\left[-\frac{1}{2}\left(\frac{\kappa_1^2}{\kappa_M^2 q^2} + \frac{\kappa_2^2}{\kappa_m^2} \right) \right]}{\pi\, q\, \kappa_M \kappa_m (1 + 2a_1 H_0 \kappa_m^2 + \kappa_1^2 H_0^2/q^2)^{1/2}} \times$$

$$\times \sin^2\left[\frac{L}{2k} \left(\frac{\kappa_1^2}{q} + \kappa_2^2 \right) \right]. \tag{13}$$

To simplify further analysis we assume that the vertical scale always satisfies the conditions

$$\kappa_M H_0 \gg 1, \quad \kappa_M^2 L k^{-1} q^{-1} \ll 1 \tag{14}$$

The meaning of the first inequality (14) is clear, while the second one enables us to deal with the spatial frequency range which is of particular interest, and at the same time to neglect the diffraction, thus, simplifying further analysis. In addition, we shall consider inhomogeneities which are isotropic or extend over the Earth's surface

$$\kappa_M \geq \kappa_m \tag{15}$$

We will characterize the anisotropy of inhomogeneities by a parameter η which for spectrum (12) is equal to $\kappa_M/\kappa_m \geq 1$. On the basis of assumption (14) one can easily calculate the one-dimensional spectrum

$$V_I(\kappa_1) = \int F_I(\kappa_1, \kappa_2)\, d\kappa_2 \tag{16}$$

which can be found from measurements I made by an observer onboard a space platform if the star is in the plane of its orbit. For frequencies where the diffraction is not significant simple calculations give:

$$V_I(\kappa_1) = \frac{\sigma_\nu^2 \kappa_M^3}{q\sqrt{2\pi}} \times$$

$$\times \frac{L^2\bar{\Psi}^2 (q^2\tilde{\kappa}_1^4 + 2q\eta^{-2}\tilde{\kappa}_1^2 + 3\eta^4) \exp(-\tilde{\kappa}_1^2/2)}{(1 + 2a_e H_0 \kappa_m^2)^{1/2} (1 + H_0^2 \kappa_M^2 (1 + 2a_e H_0 \kappa_m^2)^{-1} \tilde{\kappa}_1^2)^{1/2}} \tag{17}$$

where $\tilde{\kappa}_1 = \kappa_1 / \kappa_M q$. Besides the parameters q, η, κ_M, and H_0 considered above, another parameter $2a_e H_0 \kappa_m$ is also introduced in Eq. (17). The meaning of this parameter is clear: it characterizes the relation between the length of the effective path of the interaction of the beam and the atmosphere $2(2H_0 a_e)^{1/2}$ and the characteristic size of the refractive index inhomogeneities over the Earth's surface $2\pi\kappa_m^{-1}$.

For $2a_e H_0 \kappa_m^2 \ll 1$ variability of the refractive index along a horizontal is insignificant when scintillations are being observed, which formally corresponds to the model of a spherically layered atmosphere when $\eta \to \infty$. For this model Eq. (17) yields a simple expression for $V_I(\kappa_1)$ when $\kappa_1 H_0 \gg 1$:

$$V_I(\kappa_1) = \frac{\sigma_\nu^2 L^2 \bar{\Psi}^2 \kappa_1^3}{(2\pi)^{1/2} H_0 \kappa_M q^2} \exp\left[-\frac{\kappa_1^2}{2\kappa_M^2 q^2} \right], \tag{18}$$

It is obvious that such extensive variations of ν which extend for hundreds of kilometers over the Earth's surface, provided that their vertical scale $2\pi\kappa_M^{-1}$ is less than the height of the homogeneous atmosphere, cannot be described at all in the approximation of a model of a random field which is statistically homogeneous in a cartesian coordinate system.

It is also interesting to investigate scintillations when extension of the refractive index inhomogeneities in the atmosphere over the Earth's surface is not so vast, i.e. $\kappa_m^2 a_e H_0 \gg 1$. Generally speaking, the description of such refractive index variations in a local volume with dimensions over the Earth's surface much less than $(2a_eH_0)^{1/2}$ but much greater than $2\pi\kappa_m^{-1}$ in the approximation of statistical homogeneity in a cartesian coordinate system does not give rise to any significant objections. Nevertheless, when $\kappa_m^2 a_e H_0 \gg 1$, the effect of atmospheric sphericity can also appear because in this case the spectrum[17] strongly depends on the ratio between the two large parameters

$$\kappa_M^2 H_0^2 / 2a_e H_0 \kappa_m^2 = \eta^2 H_0 / 2a_e. \tag{19}$$

For the Earth's atmosphere $H_0 \simeq 6 \div 8$ km while $a_e = 6.4\times10^3$ km, therefore $H_0/2a_e \ll 1$, but, parameter (19) depending on the degree of anisotropy can be both small and large compared with unity.

Taking spectrum (17) as an example, let us investigate the manifestation of atmospheric sphericity in the spectra of star scintillations observed from space through the Earth's atmosphere in the case when the refractive index inhomogeneities are not so greatly extended over the Earth's surface $\kappa_m^2 a_e H_0 \gg 1$. It is quite evident that for statistically isotropic fluctuations $\eta = 1$ the sphericity of the atmosphere is manifested only through variations of the average parameters, wherefore the cases in which $\eta > 1$ should be interesting. For spectrum (12) under the conditions expressed by Eq. (14) the frequencies $\tilde{\kappa}_1$ which do not differ significantly from unity are of great interest because for high frequencies the spectral density $V_I(\kappa_1)$, decreases exponentially in all the cases. In this frequency range spectrum (17) coincides with that of Eq. (18) for $\tilde{\kappa}_1^2 \gg 2a_e / H_0\eta^2$ if ratio (19) is large, i.e. $\eta^2 \gg 2a_e/H_0$. If the opposite situation obtains i. e., $1 \le \eta^2 \ll 2a_e/H_0$, one can easily show that spectrum (17) for $\kappa_1^2 \ll 2a_e / H_0\eta^2$ does not differ from the scintillation spectrum computed on the assumption of statistical homogeneity of the field ν in the cartesian coordinate system. Thus, the assumption that the refractive index fluctuations are a statistically homogeneous field on the sphere is essential for the case of great anisotropic inhomogeneities with anisotropy parameter $\eta^2 \gg a_e/H_0$.

This conclusion can be given a descriptive physical interpretation. Anisotropic inhomogeneities are extended along the beam within its perigee when $\eta > 1$. However, due to the sphericity of the atmosphere the farther inhomogeneities (along the beam) are from the beam perigee, the larger is their slope angle towards the beam. The influence of inhomogeneities on-, a light wave due to their bending with respect to the beam become markedly weaker when the slope angle exceeds the ratio between the vertical and horizontal dimensions, i.e.. it exceeds the value η^{-1}. Since the slope angle is $\sqrt{2(h-h_p)/a_e}$ where h is the height of the point on the beam at which this inhomogeneity occurs the region of effective interactions of the light wave with the refractive index inhomogeneities is limited by fall off of the air density with altitude when $1 \le \eta^2 \ll a_e H_0^{-1}$ and by the bending of the greatly extended inhomogeneities relative to the beam in the opposite case when $\eta^2 \gg a_e/H_0$.

Along with spectrum (12) we will consider an anisotropic power spectrum

$$\Phi_\nu = A_\mu C_\nu^2 \eta^2(\kappa_1^2 + \eta^2(\kappa_2^2 + \kappa_3^2))^{-\mu/2}, \quad 3 < \mu < 5, \quad \eta \ge 1, \tag{20}$$

where C_ν^2 is the structure characteristic, and

$$A_\mu = (4\pi^2)^{-1}\Gamma(\mu-1)\sin(\pi(\mu-3)/2).$$

For expression (20) the structure function D_ν is given by

$$D_\nu(h_2-h_1, 0) = C_\nu^2 |h_2-h_1|^{\mu-3},$$

$$D_\nu(0, a_e\theta) = C_\nu^2 (a_e\theta/\eta)^{\mu-3} \tag{21}$$

The calculation of $F_I(\kappa_1, \kappa_2)$ for power spectrum (20) gives the following expression

$$F_I(\kappa_1,\kappa_2) = \frac{4\pi^{1/2}A_\mu k^2 C_\nu^2 \bar{\Psi}^2}{q(aH_0)^{1/2}} \sin^2\left[\frac{L}{2k}\left(\frac{\kappa_1^2}{q} + \kappa_2^2\right)\right] \times$$

$$\times \left[\frac{\kappa_1^2}{q^2} + \eta^2\kappa_2^2\right]^{-\mu/2} Z^{1/2} U\left[1/2, -(\mu-3)/2; Z\right], \tag{22}$$

where

$$Z = \eta^{-2} aH_0(\kappa_1^2 q^{-2} + \eta^2\kappa_2^2)(1 + \kappa_1^2 H_0 q^{-2})^{-1},$$

and $U(1/2, -(\mu-3)/2; Z)$ is a degenerate hypergeometric function in the notation of Ref. 13. For the spectrum $V_I(\kappa_1)$ and sufficiently high frequencies $\kappa_1 H_0 \gg 1$ we obtain

$$V_I(\kappa_1) = \frac{4\pi^{1/2}A_\mu k^2 C_\nu^2 \bar{\Psi}^2 q^{\mu-2}\kappa_1^{-\mu+1}}{H_0} \int_0^\infty \frac{dy}{y^{1/2}} (1+y)^{-\frac{\mu-2}{2}} \times$$

$$\times \sin^2\left[\frac{\kappa_1^2 L}{2kq}\left[1 + \frac{y}{\eta q}\right]\right] U(1/2, -(\mu-3)/2; (1+y)a_e/\eta^2 H_0). \tag{23}$$

The parameter $a_e/H_0\eta^2$, used in this expression as well as in Eq. (22), explicitly characterizes the influence of atmospheric sphericity.

For inhomogeneity spectrum (19) the peculiarities associated with atmospheric sphericity are manifested in the variance of the fluctuations σ_I^2. Integrating Eq. (22) by parts we obtain for large values of $\eta^2 > a_e/H_0$ an approximate expression

$$\sigma_I^2 = \frac{2\pi^2 A_\mu k^2 \bar{\Psi}^2 C_\nu^2 H_0^{-1}}{(\mu-2)\sin(\pi(\mu-2)/4)\Gamma(\mu/2)} \left(\frac{qL}{k}\right)^{\mu/2-1}. \quad (24)$$

In the opposite case when $1 \ll \eta^2 \ll a_e/H_0$ we obtain an approximate expression which is the same both for fluctuations statistically homogeneous on the sphere and for statistically homogeneous fluctuations.

$$\sigma_I^2 = \frac{\pi^{5/2} A_\mu k^2 \bar{\Psi}^2 C_\nu^2 \eta (H_0 a_e)^{-1/2}}{\sin(\pi(\mu-2)/4)\Gamma(\mu/2)} \left(\frac{qL}{k}\right)^{\mu/2-1}. \quad (25)$$

Comparing Eqs. (24) and (25), one can easily show that the variances of the intensity fluctuations for values of η which are not too large grow with increasing degree of anisotropy, all other conditions being equal. However, the sphericity of the atmosphere limits the increase of the variance σ_I^2 when the anisotropy becomes sufficiently strong $\eta^2 \geq a_e/H_0$.

The preceding investigation of scintillation spectra shows that in the case of anisotropic fluctuations of the refractive index in the atmosphere data analysis of extraterrestrial sources from space platforms through the Earth's atmosphere requires use of the model in which the fluctuations are statistically locally-homogeneous on the sphere.

For such a model general expressions have been derived in this paper to calculate the spectra of weak scintillations for prescribed spatial spectra of the inhomogeneities. For the concrete examples the calculations lead to simple formulas which enable us to elucidate the role of anisotropy and to show that the parameter $\eta^2 H_0/a_e$ is the criterion for deciding when it is necessary to take into account the sphericity of the atmosphere in the analysis of the data of scintillation observations.

REFERENCES

1. W.B. Hubbard, E. Lelloch, B. Sicardy, et al., Astrophysical J., **325**, No. 1, Part 1, 490 (1988).
2. R.G. French, J.L. Elliot and B. Sicardy, ICARUS, **51**, No. 3, 491 (1982).
3. R. Woo, J.W. Armstrong and A.J. Kliore, ICARUS, **52**, No. 2, 335 (1982).
4. A.S. Gurvich, Izvestiya Akad. Nauk SSSR, Fizika Atmosfery i Okeana, **5**, No. 11, 1172 (1969).
5. G.M. Grechko, A.S. Gurvich, and Yu.V. Romanenko, Izvestiya Akad. Nauk SSSR, Fizika Atmosfery i Okeana, **16**, No. 4, 339 (1980).
6. A.S. Gurvich, S.V. Zagoruiko, V. Kan, et al., Doklady Akad. Nauk SSSR, **259**, No. 6, 1330(1981).
7. W.B. Hubbard, J.R. Jokipii and A. Wilking, ICARUS, **34**, No. 2, 374 (1978).
8. R. Woo, A. Ishimaru, and F.-Ch. Yung Radio Science, **15**, No. 3, 695 (1980).
9. G.M. Grechko, A.S. Gurvich, V.A. Liakhov, et al., Izvestiya Akad. Nauk SSSR, Ser. Fizika Atmosfery i Okeana, **17**, No. 11, 1123 (1981).
10. A.M. Obukhov, Uspekhi Mat. Nauk, **2**, No. 2 (18), 196 (1947).
11. A.S. Gurvich, Izvestiya Akad. Nauk SSSR, Ser. Radiofizika, **27**, No. 8, 951 (1984).
12. V.I. Tatarskii, *Wave Propagation in the Turbulent Atmosphere* (Nauka, Moscow, 1967).
13. M.I. Abramowitz and I.M. Stegun, *Handbook of Mathematical Functions* (U. S. Govt. Printing Office, Washington, 1972).