NUMERICAL AND ASYMPTOTIC ESTIMATES OF THE INFLUENCE OF A STOCHASTIC MEDIUM MODEL DIMENSIONALITY ON THE RADIATION TRANSFER CALCULATIONS

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Difference in integral characteristics of the radiation field calculated using one- and three- dimensional models of a stochastic medium has been studied using approximate asymptotic estimates and Monte-Carlo calculations. For this purpose, standard weighting estimates obtained by Monte-Carlo method are partially averaged analytically over the distributions of the extinction coefficients in a special "Poisson" models of random media. For testing problem with the Henyey-Greenstein scattering phase function, asymptotic and numerical estimates of the transmittance probability obtained were in a close agreement. Moreover introduction of additional horizontal stochastisity decreased both the transmittance and albedo probabilities.

1. INTRODUCTION

The process of radiation transfer through a plane horizontal layer of a medium which density is a uniform random function of coordinates is considered. Radiation flux with the "unit power" incident along vertical direction on the upper boundary of the layer is assumed to be a source. In a one-dimensional model the medium density is a random function of height and it is estimated by vertical Poisson point flux with the intensity λ. The density between the flux points is assumed to be constant. It is known (see Ref. 1) that correlation length for such a random function is equal to $1/\lambda$.

Previous investigations (see, for example, Ref. 1) show that in the case of optically thick one-dimensional random layers the radiation transfer probability is mainly determined by the correlation length and the mean density. This happens because at a sufficient scattering anisotropy the transfer probability is well approximated by the asymptotic $\exp(-\tau/L)$, whereas fluctuations of the random optical thickness τ are determined by the integral parameters of random density. Thus, the one-dimensional random medium model considered is reasonably universal in the radiation transfer investigations.

To introduce a three-dimensional model similar randomization of the density in horizontal layers of vertical partioning is made. Because of the necessity of limiting horizontal point Poisson fluxes the problem of estimating the transfer probability and albedo of a particle in the case of a parallelepiped-shaped medium, or, what is the same, for a finite plane horizontal layer with reasonably large extension is considered in our numerical simulations. In a one-dimensional model of the medium the Poisson flux of points along vertical axis is constructed within the parallelepiped boundaries. The parallelepiped is split into random layers. Then the density value corresponding to a given one-dimensional function is chosen independently in each layer. In a three-dimensional model the Poisson fluxes are constructed along three coordinate axes within the parallelepiped boundaries and the density value corresponding to the one-dimensional distribution function is chosen independently in each unit parallelepiped. Partial analytical averaging of the Monte-Carlo weighting estimate over random density distribution is made at a fixed splitting. Conditions for the variance finiteness of these partially averaged estimates are formulated. Calculations of both transfer and albedo probabilities are performed.

New algorithm allows us to calculate small changes in the transfer process parameters, for example, albedo changes, when coming from a determinant medium to a stochastic one with the same average density. Besides, this algorithm makes it possible to calculate the changes in the transfer process parameters when coming from one-dimensional random field model to a three-dimensional one. The calculations performed for testing the technique using Henyey-Greenstein scattering phase function give estimate of the penetration probability changes coinciding very closely with the asymptotic one. It is found that by introducing additional horizontal stochastisity into the problem considered decreases both the transmittance probability and albedo.

2. ESTIMATE OF UNIFORM RANDOM FIELDS BASED ON POINT FLUXES

Let us consider in a three-dimensional space the following models of uniform random fields.

1) The Poisson flux of points $\tau_0 = 0$, $\tau_{i+1} = \tau_i + \Delta\tau_i$, is constructed along vertical axis x in the layer $0 \le x \le H$, $(y, z) \in R^2$. Here $\Delta\tau_i$ is the random

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quantity distributed with the probability density $\lambda_{\exp}(-\lambda t)$, $\lambda = L^{-1}$. Thus we have a division of the layer $\{0 \le x \le H,\ (y, z) \in R^2\}$ into m random layers $\tau_i \le x \le \tau_{i+1}$, $\tau_0 = 0$, $\tau_m = H$. Then, σ_i value is independently chosen in each of these layers, where

$$\sigma_i = \begin{cases} \sigma^{(1)} & \text{with the probability } p, \\ \sigma^{(2)} & \text{with the probability } 1 - p. \end{cases}$$

Normalized correlation function for this field is $\exp(-\lambda x)$ (see Ref. 1) and, hence, the correlation length is equal to $1/\lambda$.

2) Fluxes of points of the above kind are independently constructed along each axis at $0 \le x \le H_1$, $0 \le y,\ z \le H_2$. As a result, we obtain the following subdivisions:

along axis x: (τ_i, τ_{i+1}), $i = 0, 1, \dots, m_x$- layers;

along axes y: (t_j, t_{j+1}), $j = 0, 1, \dots, m_y$- layers;

along axes z: (l_k, l_{k+1}), $k = 0, 1, \dots, m_z$- layers.

Combining these three subdivisions we obtain a subdivision of the parallelepiped $0 \le x \le H_1$, $0 \le y,\ z \le H_2$ into $m_x\, m_y\, m_z$ parallelepipeds. Then the value

$$\sigma_{ijk} = \begin{cases} \sigma^{(1)} & \text{with the probability } p, \\ \sigma^{(2)} & \text{with the probability } 1 - p. \end{cases}$$

is independently chosen in each elementary parallelepiped.

These models of the fields will then be applied to solve the transfer theory problem. In fact, one can consider more general models of random fields by performing the above random division along the coordinate axes and choosing, independently in each layer (or in each parallelepiped), random value σ_i (σ_{ijk}) which corresponds to a given one-dimensional distribution function $F_\xi(x)$.

3. ASYMPTOTIC ESTIMATES OF THE PENETRATION PROBABILITY USING EQUATIONS OF THE RECONSTRUCTION THEORY

It should be pointed out that in the case of a determinate plane-parallel medium the required penetration probability $I(H)$ may be quite satisfactorily estimated by the following asymptotic formula:

$$I_{\text{as}}(\tau(H)) \asymp e^{-\tau(H)/L}, \quad \tau(H) = \int_0^H \sigma(x)\,\mathrm{d}x, \tag{3.1}$$

where L is the diffusion length. If the scattering is highly anisotropic, L can be estimated using the "transport" approximation for the scattering phase function

$$w(\nu, \nu') = (1 - \mu_0)\,(1/4\pi) + \mu_0\,\delta(\nu - \nu'), \tag{3.2}$$

This approximation conserves average cosine, μ_0, of the scattering angle. The approximate value of L being defined, in this case, by the following expressions:

$$\frac{2\tilde{l}}{\tilde{q}L} = \ln \frac{L + \tilde{l}}{L - \tilde{l}}\,; \quad \tilde{q} = \frac{q(1 - \mu_0)}{1 - q\mu_0}\,; \quad \tilde{l} = \frac{l}{1 - q\mu_0}\,.$$

As an example, the estimate $I(20) \approx 0.0236$ was obtained by Monte-Carlo method using the radiation model with standard Henyey-Greenstein scattering phase function considered in Part 5 and parameters $\mu_0 = q = 0.9$, while calculation by Eq. (3.1) with $L = 5.4$ gave $I_{\text{as}}(20) = 0.0246$. It is clear that approximation by Eq. 3.1 can be improved at high $\tau(H)$ by entering the coefficient $0.0236/0.0246 = 0.959$ in front of the exponent.

Based on the reconstruction theory using Eq. (3.1) the following asymptotic formula for the one-dimensional Poisson field $\sigma(x)$ with the parameter λ (or for the first variant from Part 2) was obtained in Ref. 1:

$$EI[\tau(H)] = EI_{\text{as}}[\tau(H)] \asymp \lambda^{-2}/[E\,(\lambda - \alpha + \sigma/L)^2]\ e^{-\alpha H}, \tag{3.3}$$

while α is calculated by the following equation:

$$\lambda\, E\,(\lambda - \alpha + \sigma/L)^{-1} = 1.$$

Here the averaging is done over one-dimensional field distribution. In this case for the binary distribution, which is the basic in this paper, one can write the following expression:

$$E\,(\lambda - \alpha + \sigma/L)^{-1} = $$
$$= p\,(\lambda - \alpha + \sigma^{(1)}/L) + (1 - p)\,(\lambda - \alpha + \sigma^{(2)}/L).$$

Now let us consider some possibilities of making similar asymptotic estimates for a three-dimensional field $\sigma(x, y, z)$ presented in Part 2. It is clear that the large-scale horizontal (along (y, z)) inhomogeneities only weakly effect on the asymptotic and therefore Eq. (3.3) with $\lambda = \lambda_x$ has also to give satisfactory results in the three-dimensional case at $\lambda_y^{-1}, \lambda_z^{-1} \gg L$.

From the other hand, it is well known that in the case of small-scale horizontal inhomogeneities (at $\lambda_y^{-1}, \lambda_z^{-1} \ll L$) horizontal averaging is admissible, except for exactly vertical directions. In other words the particle trajectory is constructed actually inside the medium with $\sigma \equiv E\sigma$.

Using transport approximation (3.2) one can consider the transfer problem in the case of isotropic scattering using the following parameters:

$$\sigma_{\text{tr}} = \nu\,\sigma, \quad \sigma_{\text{s,tr}} = q\sigma\,(1 - \mu_0), \quad q_{\text{tr}} = q\,(1 - \mu_0)/\nu,$$
$$\nu = 1 - q\mu_0.$$

Once the particle falls along vertical direction on the upper surface of the layer at the point $(x_0 = 0, y_0, z_0)$ (axis x is assumed to look downward), it can experience according to Eq. (3.2) some "delta-scattering" and then the particle either escapes through the lower layer surface $x = H$ or is absorbed or is scattered isotropically. The functional to be found, in this case or the escape probability, consists of two parts:

$$I^{(0)}(H) = I_1^{(0)}(H) + I_2^{(0)}(H), \tag{3.4}$$

where $I_1^{(0)}(H)$ is the escape probability without isotropic scattering and absorption, while

$$I_2^{(0)}(H) = \int_0^H E\, f(t, \sigma)\, i^{(0)}(t, H)\, dt,$$

where $f(t; \sigma)$ is the probability density distribution of the first isotropic scattering at a given σ, $i^{(0)}(t, H)$ is the contribution into the functional sought, i.e., the escape probability at the point $x = t$, $y = y_0$, $z = z_0$ for isotropic unit source of particles at $\sigma \equiv E\sigma$. Besides, the weighting function $E\, f(t, \sigma)$ can be approximated here by

$$f(t, E\sigma) = E\sigma_{s,tr} \exp(-\, t\, E\sigma_{s,tr}),$$

or by the distribution function of the first isotropic scattering in a modified medium with $\sigma \equiv \nu E\sigma$. As a result, based on the above discussed action of small-scale inhomogeneities on the particle transfer one can obtain the following estimate:

$$I_2^{(0)} \approx I_s^{(0)} - e^{-\nu H E\sigma},$$

where $I_s^{(0)}$ is the penetration probability through the layer with $\sigma \equiv E\sigma$ in the transport approximation. Next, using asymptotic formula for $I_s^{(0)}$ the following expression can be obtained:

$$I_2^{(0)} \approx C\, e^{-\nu H E\sigma / L_0} - e^{-\nu H E\sigma},$$

where the coefficient C is close to unity and may be determined in the manner discussed early. As an example, for the Henyey-Greenstein scattering phase function (see Part 5) we have $\nu = 0.19$, $q_{tr} = 0.09/0.19 \approx 0.4737$, $C = 0.959$ and $L_0 = 1.034$. That means that

$$I_2^{(0)} = 0.959\, e^{-0.1838H} - e^{-0.19H}.$$

It should be noted, first of all, that in order to determine $I_1^{(0)}(H)$ in the transport approximation, it is expressed by the following expression:

$$I_1^{(0)}(H) = E\, e^{-\tau_{tr}^{(1)}(H)}, \tag{3.5}$$

where

$$\tau_{tr}^{(1)}(H) = \int_0^H [\sigma_{s,tr}(T) + \sigma_{s,tr}(t)]\, dt = \int_0^H \nu\sigma(t)\, dt.$$

Thus, $I_1^{(0)}(H)$ can be asymptotically estimated using Eq. (3.3) with L replaced by ν^{-1}. As an example, let us consider the problem already described above (see also Part 5) at $\lambda^{-1} = L = 5.4$, $H = 20$, $p = 0.5$, $\sigma^{(1)} = 0.6$, $\sigma^{(2)} = 1.4$, $E\sigma = 1$.

Eq. (3.3) in the transport approximation or, what is the same, at $L = 1.034/0.19$, takes the form

$$EI(\tau(H)) \asymp 0.8915\, e^{-0.1582H}.$$

Control calculations performed by Monte-Carlo method show that it is appropriate to introduce into the latter expression an additional factor which is approximately 1.035. Thus the refined asymptotic formula is as follows:

$$EI(\tau(H)) \asymp 0.9227\, e^{-0.1582H}, \tag{3.6}$$

while at $H = 20$ $EI(\tau(20)) \approx 0.03899$.

Expression (3.3) gives the following estimate of the parameter defined by Eq. (3.5):

$$I_1^{(0)}(H) \asymp 0.8867\, e^{-0.1628H}.$$

Thus, the final estimate of $I^{(0)}(H)$ takes the following form:

$$I^{(0)}(H) \approx 0.959\, e^{-0.1838H} - e^{-0.19H} + 0.8867\, e^{-0.1628H}$$

and $I^{(0)}(20) \approx 0.03610$, that means that the particle penetration probability for the stochastic three-dimensional layer with small-scale horizontal inhomogeneities is 7% lower than that for a stochastic horizontally uniform layer.

4.PARTIALLY AVERAGED ESTIMATES OBTAINED BY MONTE-CARLO METHOD

1. In our discussion below we use the following designations: $r = (x, y, z)$, $X = (r, \rho)$ is the point of collision in the phase space of coordinates and directions.

Let us consider the random parameter (see Ref. 2):

$$\xi = Q_N(\sigma) = \prod_{i=1}^{N} \frac{k(X_{i-1}, X_i; \sigma)}{p(X_{i-1}, X_i)} D, \quad Q_0 = 1,$$

where $k(X', X; \sigma)$ is the kernel of the particle transfer integral equation (see Ref. 2) with the following parameters:

$$\sigma_s(r) = q\, \sigma(r), \quad \sigma_c(r) = (1 - q)\, \sigma(r), \quad 0 < q < 1, \tag{4.1}$$

where q is the particle survival probability at a collision, or, what is the same, the scattering probability at a scattering phase function $w_s(\mu)$; μ is the cosine of the scattering angle. The kernel with the following parameters:

$$\sigma_{s,0} = q_0\,\sigma_0\ , \quad \sigma_{c,0} = (1-q)\,\sigma_0\ , \quad 0< q_0 < 1 \qquad (4.2)$$

and the same scattering phase function is considered as the transition density $p(X', X)$ of the Markov chain of collisions to be estimated. The value of D is equal to zero or unity in accordance with the escape variant whose probability is estimated. For this particular $\sigma(r)$, the weighting factor $k(X_{i-1}, X_i; \sigma)/p(X_{i-1}, X_i)$ represents the ratio between the probability densities of free path length after scattering at the point X_{i-1} which are calculated for the medium models with the parameters given by Eqs. (4.2) and (4.3), and a given scattering phase function (see Ref. 2). The probability sought

$$P = E_\sigma\, M_\omega\, \{Q_N(\sigma)\,|\,\sigma\}$$

represents the average value of the functional to be estimated for random medium (ω is the random trajectory) (see Ref. 2).

For the first model of the random field presented in Part 2 one can write the following expression:

$$\overline{p} = M_{\omega,\{\tau_i\}}\left\{\prod_{j=1}^{m}\left(p\left(\frac{q\sigma^{(1)}}{q_0\,\sigma_0}\right)^{n_i}\exp\left(-\left(\sigma^{(1)}-\sigma_0\right)l_i\right) + \right.\right.$$

$$\left.\left. + (1-p)\left(\frac{q\sigma^{(2)}}{q_0\,\sigma_0}\right)^{n_i}\exp\left(-\left(\sigma^{(2)}-\sigma_0\right)l_i\right)\right)\right\}, \qquad (4.3)$$

where m is the number of random layers along the x axis, σ_i is the random value of σ in the ith layer, n_i is the number of the particle collisions in the ith layer, l_i is the particle free path in the ith layer, N is the random number of the last state of the collision chain and thus x_N is the absorption point or the first collision point out of the area which is assumed to be surrounded by fictitious medium, in which $\sigma = \sigma_0$ and $q = 0$.

For the second field model Eq. (4.3) takes the following form:

$$E_\sigma\, M_\omega\,\{Q_N(\sigma)\,|\,\sigma\} = M_{\omega,\{\tau_i\},\{t_j\},\{l_k\}}\times$$

$$\times\left\{\prod_{i=1}^{m_x}\prod_{j=1}^{m_y}\prod_{k=1}^{m_z}\left(p\left(\frac{q\sigma^{(1)}}{q_0\,\sigma_0}\right)^{n_{ijk}}\exp\left(-\left(\sigma^{(1)}-\sigma_0\right)l_{ijk}\right)+\right.\right.$$

$$\left.\left.+(1-p)\left(\frac{q\sigma^{(2)}}{q_0\,\sigma_0}\right)^{n_{ijk}}\exp\left(-\left(\sigma^{(2)}-\sigma_0\right)l_{ijk}\right)\right)\right\}, \qquad (4.4)$$

where n_{ijk} is the number of particle collisions in the ijkth parallelepiped, l_{ijk} is the free path of a particle within this parallelepiped, m_x, m_y, and m_z are the numbers of layers along the x, y and z axis, respectively.

As mentioned above, the distribution of σ_i (σ_{ijk}) does not need to be Bernulli one. It may have any distribution function $F_\xi(x)$. Moreover, if we manage to express analytically the mathematical expectation $E_\sigma\{Q_N\ (\sigma)\,|\,\omega,\{\tau_i\}\}$ at fixed trajectories and layer boundaries, expressions similar to Eqs. (4.3) and (4.4) are obtained. But if the integral representing the expectation

$$\int_R\left(\frac{q\sigma}{q_0\,\sigma_0}\right)^{n_i}\exp\left(-(\sigma-\sigma_0)\,l_i\right)\mathrm{d}F_\xi(\sigma)$$

can not be calculated analytically, it may be replaced by some approximate quadrature formula.

2. Then, the conditions for finiteness of the variance of partially averaged weighted estimate are deduced.

The variance is finite if the parameter $E_\sigma\, M_\omega\,\{Q_N^2(\sigma)\,|\,\sigma\}$, is finite, where

$$Q_N(\sigma) = \prod\left(\frac{q\sigma}{q_0\,\sigma_0}\right)^{n}\exp\left(-(\sigma-\sigma_0)\,l\right).$$

Here σ is the random density value in ijkth parallelepiped, n and l are the number of the particle collisions and the free path in ijkth perallelepiped, respectively.

Let us consider the following expression:

$$Q_N^2(\sigma) = \prod\left(\frac{q^2\sigma^2}{q_0^2\,\sigma_0^2}\right)^{n}\exp\left(-2\,(\sigma-\sigma_0)\,l\right) =$$

$$=\prod\left(\frac{\overline{\sigma}}{q_0\,\sigma_0}\right)^{n}\frac{\exp\left(-(\overline{\sigma}-\sigma_0)\,l\right)}{\exp\left(-(\overline{\sigma}-\sigma_0)\,l\right)}\exp\left(-2\,(\sigma-\sigma_0)\,l\right),$$

where $\overline{\sigma} = q^2\,\sigma^2/(q_0\,\sigma_0)$.

If $\exp(-2(\sigma-\sigma_0)l)/\exp(-(\overline{\sigma}-\sigma_0)l)$ does not exceed l, the average value of $Q_N^2(\sigma)$ is not higher than that of the functional to be estimated for the cross section $\overline{s}$ which is finite.

Thus, the inequality $\exp\left(-2\,(\sigma-\sigma_0)\,l\right) + (\overline{\sigma}-\sigma_0)\,l) \le 1$ should be solved to determine σ. This inequality is equivalent either to

$$2\,(\sigma_0-\sigma) + (\overline{\sigma}-\sigma_0) \le 0$$

or

$$\sigma^2 - 2\sigma + q^2 \le 0 \qquad (4.5)$$

if we assume that $q_0 = 1$, $\sigma_0 = q^2$.

3. Now let us consider the geometric part of the modeling algorithm. First, we construct a division of a given parallelepiped into random unit parallelepipeds for each trajectory. Then the following algorithm of the free path l_{ijk} calculations within these parallelepipeds is used.

It is assumed that l is the free path (chosen according to σ_0); (nx, ny, nz) is the number of the parallelepiped where the next collision occurs, $(nx1, ny1, nz1)$ is the number of the parallelepiped where previous collision took place. First we calculate, in each layer, the parts of the free path $lx[nx1], ..., lx[nx]$, $ly[ny1], ..., ly[ny]$; $lz[nz1], ...,$ $lz[nz]$, directed along x, and y, z axis, respectively. The sum of the parts directed along each axis is equal to l.

Then, current values $nx1$, $ny1$, $nz1$ are assigned to the indices i, j, k, respectively and values of $lx[i]$, $ly[j]$, $lz[k]$ are compared. If $lx[i]$ is the least among them, current value of $l[i, j]\,[k]$ is increased by $lx[i]$. Current value of the index I is also changed in the following manner: if $nx1$ exceeds $nx1$, i is increased by unity, and if nx is lower than $nx1$, i is decreased by unity. If $ly[j]$ or $lz[k]$ are the lowest among, $lx[i]$, $ly[j]$, $lz[k]$, current value of $l[i, j]\,[k]$ is increased by $ly[j]$ or by $lz[k]$, respectively. Current values of indices j and k are also changed. But in the case when $nx1 = nx$, $ny1 = ny$, $nz1 = nz$ $l[i, j]\,[k]$ is increased by l. This process terminates when $i = nx$, $j = ny$, $k = nz$.

5. TEST TASK

Now consider the problem on estimating of the average penetration probability and average backscatter (albedo) in a medium layer $0 \le x \le H_1$, $0 \le y, z \le H_2$, which density is the random field described in Part 2, while $p = 0.5$, $\sigma^{(1)} = 0.6$, $\sigma^{(2)} = 1.4$, $q = 0.9$. The problem is solved by estimating the particle trajectory using standard techniques (see Ref. 2). The value of the scattering angle cosine is estimated according to the Henyey-Greenstein scattering phase function

$$w_s(\mu) = \frac{1}{2}\frac{1 - \mu_0^2}{(1 + \mu_0^2 - 2\mu\mu_0)^{3/2}}\,; \quad -1 \le \mu \le 1,$$

$\mu_0 = E\mu = 0.9.$

For such scattering phase function at $q = 0.9$ using the weighting algorithm one can obtain $L \approx 5.4$. The particle trajectory estimates are made for a determinate medium with $\sigma_0 = 0.81$ and $q_0 = 1$. It is easy to see from Eq. (4.5) that the condition for finiteness of the variance of the Monte-Carlo averaged weighted estimate is fulfilled. The trajectories are directed from the point $(0, H_2/2, H_2/2)$ along the x axis and are observed until the moment of particle escape from the region $0 \le x \le H_1$, $0 \le y, z \le H_2$. Out of this region $q_0 = 0$. The following versions of calculations were carried out for the above discussed random medium models:

A) $\lambda_x = \lambda_y = \lambda_z = 1/L$, $H_1 = 20$, $H_2 = 100$;

B) $\lambda_x = \lambda_y = \lambda_z = 1/L$, $H_1 = 20$, $H_2 = 40$.

Total number of $N = 100000$ trajectories were used in each variant.

In what follows we use the designations: $P^{(p)}$ and $P^{(a)}$ which are the average penetration and albedo probabilities, respectively, while 3s, 1s and 1d designate three-dimensional stochastic, one-dimensional stochastic and one-dimensional determinate with the extinction coefficient $\sigma \equiv E\sigma$ media, respectively, σ_N is the estimate of the rms error in the results calculated. Let us briefly review the results obtained for the basic variant A. As is evident from Table II, the average penetration probability for the three-dimensional random field model is approximately 9% lower than that for the one-dimensional one.

TABLE I. Average penetration probability, calculations.

	$P^{(p)}_{3s}$	σ_N	$P^{(p)}_{1s}$	σ_N
A	0.0348	0.0004	0.0381	0.0004
B	0.0338	0.0003	0.0366	0.0004

TABLE II. Average penetration probability variations obtained when passing from one-dimensional stochastic medium to a three-dimensional one.

	$P^{(p)}_{3s} - P^{(p)}_{1s}$	σ_N
A	– 0.0033	0.0003
B	– 0.0027	0.0002

TABLE III. Average albedo, calculations.

	$P^{(a)}_{3s}$	σ_N	$P^{(a)}_{1s}$	σ_N	$P^{(a)}_{1d}$	σ_N
A	0.0662	0.0006	0.0676	0.0008	0.0681	0.0007
B	0.0647	0.0005	0.0653	0.0007	0.0671	0.0006

One can see from Table IV that the average albedo decreases by 2.8% in the determinate medium, as compared to a three-dimensional stochastic one with the same average density.

Table V shows that the average albedo decreases by 2.1% in the one-dimensional stochastic medium compared to a three-dimensional one.

TABLE IV. Average albedo variations due to change from determinate medium to a three-dimensional stochastic one.

	$P^{(a)}_{1d} - P^{(a)}_{3s}$	σ_N
A	0.0019	0.0004
B	0.0023	0.0003

TABLE V. Average albedo variations obtained due to change from a one-dimensional medium model to a three-dimensional one.

	$P^{(a)}_{1s} - P^{(a)}_{3s}$	σ_N
A	0.0014	0.0004
B	0.0006	0.0003

TABLE VI. The change of average albedo due to the change from determinate model to a stochastic one-dimensional one.

	$P^{(a)}_{1d} - P^{(a)}_{1s}$	σ_N
A	0.0005	0.0006
B	0.0017	0.0004

For the changes in the average albedo due to the change from a determinate medium to a one-dimensional stochastic one to be found, additional calculations were carried out in the infinite layer using N=4000000 trajectories. As a result we obtain the following estimate:

$$P^{(a)}_{1d} - P^{(a)}_{1s} \approx 0.0001.$$

These albedo changes are quite reasonable since it is obvious that in the layers with high optical thickness τ the albedo P_a, is a convex function of τ (that is, $P'_a(\tau) > 0$, $P''_a(\tau) < 0$), whose average value is lower than its value at average τ.

Note that the estimate

$$P^{(p)}_{3s} - P^{(p)}_{1s} = -0.0033$$

obtained in variant A with the approximate rms error $\sigma_N = 0.0003$ well agrees with the analytical estimate

$$I^{(0)}(20) - EI(\tau(20)) = -0.00289,$$

obtained in Part 3 for the case of low-scale horizontal inhomogeneities. This means that the model described in Part 3 is quite satisfactory even at λ_y^{-1}, $\lambda_z^{-1} \approx L$. This occurs, probably, due to the presence of inhomogeneities along both y and z axis.

As was discussed in the Introduction, it is a little bit unexpected fact that the change from a one-dimensional stochastic medium to a three-dimensional one decreases not only the penetration probability, but the albedo probability, as well.

Comparison between the results obtained in variants A and B gives the estimate of the parallelepiped horizontal size finiteness effect on the average penetration and albedo probabilities for the "centered" radiation sources. Specifically, decrease in the horizontal size considerably increases the changes in average albedo when going from a determinate medium to a stochastic one.

The models and estimates discussed above are reasonably general since Monte-Carlo calculations indicate that average radiation flux passing through the stochastic optically thick layer is mainly determined by the medium correlation length, which is equal to λ^{-1} for the Poisson field.

6. CONCLUSION

Based on the results obtained we can conclude that when performing calculations of the radiation transfer, stochastically nonuniform layer can be replaced in the first approximation by a uniform layer with the total extinction coefficient equal to $\sigma = \alpha L$ where L is the diffusion length for a given scattering phase function and survival coefficient q, while the value of α is determined by the one-dimensional Poisson field approximation according to Eq. (3.3). It should be taken into account that using this substitution for the layer with reasonably high optical thickness one can also reconstruct albedo at different levels. In the case of high optical thickness and if horizontal inhomogeneity is sufficient one may consider only first term in Eq. (3.4). Thus, in the transport approximation α is determined by the directly transmitted "delta-scattered" radiation.

The constants preaveraged in this way can be improved by considering the results obtained by Monte-Carlo method for the initial complex stochastic medium models to be experimental and then estimating the constants for simplified models by the methods of solving the parametric inverse problem (see Ref. 1).

REFERENCES

1. G.I. Martchuk, G.A. Mikhailov, M.A. Nazarliev et. al., *Monte-Carlo Method in Atmospheric Optics* (Nauka, Novosibirsk, 1976), 280 pp. [Engl. Transl. Springer-Verlag, 1980].
2. G.A.Mikhailov, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **18**, 1289–1295 (1982).