# Optical theorem for scattering medium with lens properties

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Received August 31, 2006

In this paper we present a generalization of the optical theorem to the case of radiation scattering by a particle placed into a regularly inhomogeneous medium. The generalization was made based on the energy balance analysis of incident, scattered, and absorbed radiation without using traditional assumption about a plane wave scattering in a homogeneous medium surrounding the scatterer.

The problem of scattering of a wave on a particle placed in a homogeneous medium has been studied quite well (see, for example, Refs. 1 and 2). However, in solving some problems of radiative transfer, researchers often meet situations when it is necessary to calculate scattering characteristics of a particle placed in a regularly inhomogeneous medium – a medium with lens properties. Among such problems there are, for example, investigations into the propagation of optical radiation through the channel of droplet aerosol to be cleared up,[3] thermal blooming of high-power laser beams in the atmosphere,[4] light scattering by non-equilibrium systems,[5] as well as into the ocean acoustics.[6]

Regardless that such problems are often in practice, investigations in this field are not numerous and are mainly related to calculations of differential characteristics of scattering. Nevertheless, these investigations show (see, for example, Ref. 7), that regular inhomogeneity of the medium surrounding a particle leads to essential change of the scattering characteristics of the particle, in particular, the shape of its scattering diagrams in the inhomogeneous focusing (defocusing) and homogeneous media are significantly different.

In this paper, we generalize the optical theorem to the case of wave scattering on a particle placed in a regularly inhomogeneous medium. As known,[1,2,8] the optical theorem is one of the important results of the theory of scattering, so its confirmation for the case when the scatterer has been placed in a medium with lens properties is the necessary element of the theory development. Besides, generalization of the optical theorem can be also useful for the problems, in which scattering of non-plane waves is considered, for example, the problems of levitation of particles under the effect of the laser radiation, in which the total cross sections are now being calculated directly by integration and summing of the series of scattering amplitudes.[9]

Let a particle with dielectric constant $\varepsilon_0$, on which the linearly polarized wave $\mathbf{E}_\mathrm{i}(\mathbf{R})$ is scattered, be placed into a spatially inhomogeneous transparent medium with the dielectric constant

$$\varepsilon_\mathrm{m}(\mathbf{R}) = \varepsilon_\mathrm{m}(0)\big(1 + \Delta\varepsilon_\mathrm{m}(\mathbf{R})\big),$$

where $\mathbf{R} = \{x, y, z\}$ is the radius-vector, $\Delta\varepsilon_\mathrm{m}(\mathbf{R})$ is the relative deviation of the dielectric constant of the medium from some characteristic value $\varepsilon_\mathrm{m}(0)$. Let us consider the inhomogeneities $\Delta\varepsilon_\mathrm{m}(\mathbf{R})$ which allow one to ignore the depolarization in the medium of both incident $\mathbf{E}_\mathrm{i}$ and scattered $\mathbf{E}_\mathrm{s}(\mathbf{R})$ waves and assuming the dielectric constant of the medium constant within the geometric boundaries of the particle, i.e.,

$$\Delta\varepsilon_\mathrm{m}(\mathbf{R})\big|_{|\mathbf{R}-\mathbf{R}_\mathrm{s}|\le a} \approx \Delta\varepsilon_\mathrm{m}(\mathbf{R}_\mathrm{s}),$$

where $\mathbf{R}_\mathrm{s}$ is the coordinate of the particle, $a$ is its characteristic linear size.

Let us study scattering of the wave on a big particle (its linear size significantly exceeds the wavelength of radiation), when the main fraction of energy of the scattered radiation in a homogeneous medium has been concentrated within a narrow cone of angles about the dominant direction of the incident wave propagation. At scattering on such a particle in smoothly inhomogeneous medium, the directions of propagation of the energy-carrying fractions of the scattered and incident waves do not differ strongly. So, the propagation of forward scattered and incident waves can be considered in the quasi-optical approximation.

To find the sought formula relating the extinction cross section of a particle with the strength of the electric field inside the particle and of the incident wave, let us determine, first of all, the power of radiation passed through the forward hemisphere, the center of which has been placed inside the scatterer. In the frameworks of applicability of the quasi-optical approximation, this flux is equal to the flux passed through the tangent plane to the hemisphere, the normal direction to which coincides with the dominant direction of propagation of the incident wave.

0235-6880/07/03 189-03 $02.00 

Let us present the field $\mathbf{E}(\mathbf{R})$ behind the particle (relative to the direction of propagation of the incident wave) in the form of the sum of the incident $\mathbf{E}_\mathrm{i}$ and forward scattered $\mathbf{E}_\mathrm{s}^{(+)}$ waves:

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_\mathrm{i}(\mathbf{R}) + \mathbf{E}_\mathrm{s}^{(+)}(\mathbf{R}). \qquad (1)$$

The intensity of the total field in the plane $z = \mathrm{const}$ tangent to the forward hemisphere perpendicular to the dominant direction of propagation of the incident wave and situated quite far from the scattering particle (so that the forward scattered field makes the main contribution to the intensity) is determined by the formula

$$I(\mathbf{R}_\perp,z) = I_\mathrm{i}(\mathbf{R}_\perp,z) + I_\mathrm{s}^{(+)}(\mathbf{R}_\perp,z) + \tilde{I}(\mathbf{R}_\perp,z), \qquad (2)$$

where

$$I_\mathrm{i} = cn_\mathrm{m}|E_\mathrm{i}|^2/8\pi$$

is the intensity of the incident wave;

$$I_\mathrm{s}^{(+)} = cn_\mathrm{m}\left|E_\mathrm{s}^{(+)}\right|^2/8\pi$$

is the intensity of the forward scattered wave;

$$\tilde{I} = cn_\mathrm{m}\left(E_\mathrm{i}E_\mathrm{s}^{(+)*} + E_\mathrm{i}^*E_\mathrm{s}^{(+)}\right)/8\pi$$

is the interference term, $n_\mathrm{m} = \sqrt{\varepsilon_\mathrm{m}(0)}$ is the characteristic value of the refractive index of the medium, $E_\mathrm{i}$ and $E_\mathrm{s}^{(+)}$ are the components of the amplitudes of the incident and forward scattered waves.

Integrating Eq. (2) over the cross coordinate $\mathbf{R}_\perp$ on the entire plane $z$, we obtain the formula for the power of radiation in the considered plane:

$$P(z) = P_\mathrm{i}(z) + P_\mathrm{s}^{(+)}(z) + \tilde{P}(z), \qquad (3)$$

where

$$P_\mathrm{i}(z) = \iint \mathrm{d}^2R_\perp I_\mathrm{i}(\mathbf{R}_\perp,z)$$

is the power of incident wave;

$$P_\mathrm{s}^{(+)}(z) = \iint \mathrm{d}^2R_\perp I_\mathrm{s}^{(+)}(\mathbf{R}_\perp,z)$$

is the power of the forward scattered wave;

$$\tilde{P}(z) = \iint \mathrm{d}^2R_\perp \tilde{I}(\mathbf{R}_\perp,z)$$

is the addition to the power caused by interference of the incident and forward scattered waves.

On the other hand, as the medium out of the particle is assumed transparent, the deficiency of the energy of radiation is caused by absorption by the particle $P_\mathrm{a}$ and the back scattering $P_\mathrm{s}^{(-)}$. Hence, one can write for the power of radiation in the plane $z$:

$$P(z) = P_\mathrm{i}(z) - P_\mathrm{a} - P_\mathrm{s}^{(-)}.$$

Then Eq. (3) will take the form:

$$P_\mathrm{a} + P_\mathrm{s} = -\tilde{P}(z), \qquad (4)$$

where $P_\mathrm{s} = P_\mathrm{s}^{(+)}(z) + P_\mathrm{s}^{(-)}$ is the total power of the scattered radiation.

As we assumed that the medium is smoothly inhomogeneous, one can describe the incident and forward scattered waves in the quasi-optical approximation:

$$E_\mathrm{i}(\mathbf{R}_\perp,z) = \iint \mathrm{d}^2\rho' E_0(\boldsymbol{\rho}')G(\mathbf{R}_\perp,z|\boldsymbol{\rho}',0)\exp(ikz),$$

$$E_\mathrm{s}(\mathbf{R}_\perp,z) =$$

$$= -i\frac{k}{2}\iiint \mathrm{d}^3r\Delta\varepsilon(\mathbf{r})E(\mathbf{r})G(\mathbf{R}_\perp,z|\mathbf{r})\exp[ik(z-r_z)], \qquad (5)$$

where

$$\Delta\varepsilon(\mathbf{r}) = \left(\varepsilon_0 - \varepsilon_\mathrm{m}(\mathbf{R}_\mathrm{s})\right)\Theta(\mathbf{r});$$

$$\Theta(\mathbf{r}) = \begin{cases} 1 & \text{inside the particle,} \\ 0 & \text{outside the paticle;} \end{cases}$$

$k = \dfrac{\omega}{c}n_\mathrm{m}$ is the wave number in the medium; $E(\mathbf{r})$ is the strength of the electric field inside the particle; $r_z$ is the $z$-component of the variable of integration; $G(\mathbf{R}|\mathbf{R}')$ is the Green function of the parabolic equation

$$2ik\frac{\partial}{\partial z}G + \Delta_\perp G + k^2\Delta\varepsilon_\mathrm{m}(\mathbf{R})G = 0; \qquad (6)$$

$$G(\mathbf{R}|\mathbf{R}')\big|_{R_z=R'_z} = \delta(\mathbf{R}_\perp - \mathbf{R}'_\perp).$$

Substituting Eq. (5) into Eqs. (2) and (3) and taking into account the group property of the Green function

$$\iint \mathrm{d}^2R_\perp G(\mathbf{R}_\perp,z|\mathbf{R}_1)G^*(\mathbf{R}_\perp,z|\mathbf{R}_2) = \delta(\mathbf{R}_{\perp1} - \mathbf{R}_{\perp2}),$$

at $z_1 \ge z_2$, let us transform Eq. (4) to the form

$$P_\mathrm{a} + P_\mathrm{s} = \frac{c}{8\pi}n_\mathrm{m}k\,\mathrm{Im}\iiint \mathrm{d}^3r\Delta\varepsilon(\mathbf{r})E(\mathbf{r})E_\mathrm{i}^*(\mathbf{r}). \qquad (7)$$

In order to obtain the formula for the total cross section of extinction based on Eq. (7), one should divide both parts of Eq. (7) by the intensity of radiation incident on the particle. One should keep in mind that, in the general case, the incident wave is not plane, and so one should make normalization to the surface-mean intensity

$$< I_\mathrm{i} >_\Sigma = \left(\iint_\Sigma \mathrm{d}SI_\mathrm{i}(\mathbf{R})\right)/\Sigma, \qquad (8)$$

where integration is done over the particle surface $\Sigma$ irradiated by the incident wave.

Based on Eq. (7) and taking into account the remark (8), we obtain the formula for the total cross section of extinction (combined from the cross sections of scattering $\sigma_s$ and absorption $\sigma_a$):

$$\sigma_t = \sigma_s + \sigma_a = \frac{c}{8\pi} n_m k \, \mathrm{Im} \frac{\iiint d^3 r \Delta\varepsilon(\mathbf{r}) E(\mathbf{r}) E_i^*(\mathbf{r})}{< I_i >_\Sigma}, \quad (9)$$

which is the generalization of the optical theorem to the case of scattering of non-plane wave on the particle placed into an inhomogeneous medium. According to the definition of the function $\Delta\varepsilon(\mathbf{r})$ (5), integration in Eq. (9) is performed over the particle volume.

Equation (9) was derived from consideration of the scalar problem. Assuming that the known relationships for a homogeneous medium[1] are fulfilled for the vectors of electric and magnetic fields in smoothly inhomogeneous medium, and taking into account that the main contribution to the integral (3) for $\tilde{P}(z)$ is determined by the forward scattered radiation, let us write Eq. (9) taking into account the vector nature of the fields:

$$\sigma_t = \frac{c}{8\pi} n_m k \, \mathrm{Im} \frac{\iiint d^3 r \Delta\varepsilon(\mathbf{r}) \mathbf{E}(\mathbf{r}) \mathbf{E}_i^*(\mathbf{r})}{< I_i >_\Sigma}. \quad (10)$$

By performing simple substitution of the relationships for the field inside the particle in the Rayleigh, Rayleigh–Gans, and eikonal approximations,[1,2] one can easily obtain the known relationships for the particle cross section of extinction in a homogeneous medium from Eq. (10).

## Acknowledgements

The work was supported in part by Russian Foundation for Basic Research and government of Kaluga region (grant No. 04-02-97232).

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