ON THE RESTORATION OF IMAGES DISTORTED BY SYMMETRIC BLURRING

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The possibility of restoration of a two-dimensional image distorted by symmetric blurring from the components of its Fourier spectrum was studied theoretically and confirmed by mathematical modeling.

In different areas of applied optics, for example, aerophotography as well as in astronomy, when objects are observed through the atmosphere with long exposure times or taking into account atmospheric dispersion, the recorded distorted image Hr), is described by a convolution integral

$$I(r) = \int J(r_1)\, H(r - r_1) dr_1, \tag{1}$$

where $J(r)$ is the true image and $H(r)$ is the blurring function. The problem of reconstructing the image $J(r)$ is usually solved by the well-developed methods of linear filtering,[1] assuming, in the process, that the blurring function is completely known. However, there often arise situations when information about the structure of the blurring is limited to the condition that the blurring function be symmetric. For example, for linearly blurred images $H(r)$ is an even function $H(-r) = H(r)$, and for defocused images it is circularly symmetric: $H(r) = H(|r|)$. These properties of $H(r)$ make it possible to extract undistorted information about the ratio of the components of the Fourier spectrum of the image and to formulate the problem of compensating for the blurring as a problem of reconstructing the image from this ratio. Indeed, Fourier transforming (1), we obtain in the region of spatial frequencies x

$$\tilde{I}(x) = \tilde{J}(x)\cdot\tilde{H}(x). \tag{2}$$

If $H(r)$ is an even function, then $\tilde{H}(x)$ is a real function, so that the equality

$$\frac{\mathrm{Im}\,\tilde{I}(x)}{\mathrm{Re}\,\tilde{I}(x)} = \frac{\mathrm{Im}\,\tilde{J}(x)}{\mathrm{Re}\,\tilde{J}(x)} \tag{3}$$

or, which is equivalent,

$$\mathrm{tg}\Psi(x) = \mathrm{tg}\Theta(x), \tag{4}$$

where $\Psi(x) = \arg\tilde{I}(x)$, $\Theta(x) = \arg\tilde{J}(x)$, is satisfied. We note that since $\mathrm{tg}\Psi(x)$ is insensitive to jumps by π, owing to the phase of the transfer function of blurring $H(x)$, $\Theta(x)$ can differ from $\Psi(x)$ by π radians. In the case of circularly symmetric blurring $H(x) = H(x_1, x_2)$ is invariant to a rotation by, for example, 90°, i.e., $H(x_1, x_2) = H(x_2, -x_1)$. For this reason we obtain from Eq. (2) a relation of the form

$$\frac{\tilde{I}(x_1, x_2)}{\tilde{I}(x_2, -x_1)} = \frac{\tilde{J}(x_1, x_2)}{\tilde{J}(x_2, -x_1)} = \alpha(x_1, x_2). \tag{5}$$

We stress that Eq. (5) is a particular case of Eq. (4), since circular symmetry of $H(x)$ is a particular case of the condition that $H(x)$ be even.

We shall study the uniqueness of image reconstruction from Eq. (4). This question was investigated in detail in Ref. 2, where it is shown that in the two-dimensional discrete case, important in practice, for a known region S of the true image (S is defined as the minimum rectangular region outside which the image disappears) the problem has, as a rule, a unique solution (in the sense of Lebesgue measure). For discrete $J(n_1, n_2)$ the analysis uniqueness reduces to analysis of the factorability of the z-transform

$$R_J(z_1, z_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} J_{n_1 n_2} z_1^{n_1} z_2^{n_2},$$

which is a two-dimensional polynomial, into a product of at least two polynomials of lower degree. In Ref. 3 it is shown that the Lebesgue measure of the set of factorable polynomials among the set of all two-dimensional polynomials is equal to zero, whence follows the conclusion that the solution is unique. We shall illustrate the method of proof of the uniqueness of reconstruction from expression (5). Assume that in addition to $J(n_1, n_2)$ these is another solution $J(n_1, n_2)$ with the same region S.

From Eq. (5) we obtain for their z-transforms

$$R_J(z_1, z_2)\, R_{J_1}(z_2, z_1^{-1}) = R_{J_1}(z_1, z_2)\, R_J(z_2, z_1^{-1}).$$

Since the polynomials are, as a rule, non-factorable, then necessarily either

$$R_J(z_1, z_2) = R_{J_1}(z_1, z_2)$$

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or

$$R_J(z_1,\ z_2) = R_J(z_2,\ z_1^{-1}).$$

For images that are not circularly symmetric the second case is excluded. For this reason only the first case is possible, whence it follows that $J(n_1, n_2) = J_1(n_1, n_2)$. Thus the problem almost always has a unique solution.

In our opinion the most promising methods for solving problems in practice are based on iterational algorithms of the form[4]

$$J_{k+1} = \hat{P}_1\hat{P}_2\hat{P}_3 J_k,$$

where J_k is the estimate of the image at the kth iteration, $\hat{P}_1$ is a projection operator projecting on the set of positive functions, $\hat{P}_2$ is a projection operator projecting on the set of finite functions (with a given region S), and $\hat{P}_3$ is the projection operator projecting on the set of functions with a fixed ratio (Eq. (4) or (5)) of the components of the Fourier spectrum. The essential feature of this approach lies in the fact that the estimate J_k approaches asymptotically the true image by means of successive matching of the image in accordance with fixed a prior restrictions. Since all the sets studied are convex, the convergence of the iteration process is guaranteed.[4]

It is worth noting that an analogous approach has been used successfully for solving similar phase and amplitude problems.[5,6] Starting from the definition of a projection, it is not difficult to prove that the action of the operator $\hat{P}_1$ reduces to setting to zero negative values, the action of the operator $\hat{P}_2$ reduces to setting to zero the values outside the given region S, and the operator $\hat{P}_3$ changes the spectrum of the estimate so that the resulting spectrum would satisfy either Eq. (4) or Eq. (5). The expression for the operator $\hat{P}_3$ in the problem of reconstruction based on the relation (4) can be derived as follows. After Fourier transforming the estimate of the image J_k we have the Fourier spectrum $\tilde{J}_k = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$, which must be corrected in the minimum possible manner so that the ratio of the new $\mathrm{Re}_1(x)$ and $\mathrm{Im}_1(x)$ would satisfy the condition $\dfrac{\mathrm{Im}_1}{\mathrm{Re}_1} = \mathrm{tg}\Theta$. From here there follows the condition

$$\sum_x |\tilde{J}_k(x) - \tilde{I}_k(x)|^2 = \sum_x \{\mathrm{Re}(x) - \mathrm{Re}_1(x)\}^2 + \{\mathrm{Im}(x) - \mathrm{Im}_1(x)\}^2 = \min.$$

Substituting into this expression $\mathrm{Im}_1 = \mathrm{Re}_1\,\mathrm{tg}\Theta$ and differentiating with respect to Re_1, we find the minimum (for given $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$). The expressions Re_1 and Im_1 have the form

$$\mathrm{Re}_1(\vec{x}) = \frac{\mathrm{Re}(x) + \mathrm{tg}\,\Theta\ \mathrm{Im}(x)}{1 + \mathrm{tg}^2\Theta},$$

$$\mathrm{Im}_1(x) = \frac{\mathrm{tg}\,\Theta}{1 + \mathrm{tg}^2\Theta}\{\mathrm{Re}(x) + \mathrm{tg}\Theta\cdot \mathrm{Im}(x)\}.$$

Writing this expression in greater detail, taking into account the fact that $\mathrm{Re}(x) = |\tilde{J}_k|\cos\varphi_k(x)$, $\mathrm{Im}(x) = |\tilde{J}_k|\sin\varphi_k(x)$, we obtain the following expression for the new Fourier spectrum

$$\tilde{I}_k = \mathrm{Re}_1(x) + i\,\mathrm{Im}_1(x):$$

$$\tilde{I}_k = |\tilde{J}_k|\cos\{\Theta(x) - \varphi_k(x)\}\exp\{i\Theta(x)\}.$$

The operator $\hat{P}_3$ is thereby determined as $\hat{P}_3 J_k = \hat{F}^{-1}\{\tilde{I}_k\}$ where $\varphi(\vec{x}) = \arg\tilde{J}_k$, $\tilde{J}_k = \hat{F}\{J_k\}$, and $\hat{F}$ and $\hat{F}^{-1}$ are, respectively, the direct and inverse Fourier transformation operators. We note that the form of the operator does not change when in is added to $\Theta(x)$; this corresponds to the initial formulation of the problem.

In deriving the operator $\hat{P}_3$ from the given ratio $\alpha(x_1, x_2)$ (5), because the Fourier spectrum is discrete it is convenient to perform the analysis for the first ($x_1, x_2 > 0$), second ($x_1 < 0$, $x_2 > 0$), third ($x_1, x_2 < 0$), and fourth ($x_1 > 0$, $x_2 < 0$) quadrants separately. We denote by $\alpha_1 = \tilde{J}(x_2)/\tilde{J}(x_1)$ the ratio of the spectrum $\tilde{J}(x_2)$ in the second quadrant ($x_1 < 0$, $x_2 > 0$) to $\tilde{J}(x_1)$ in the first quadrant ($x_1 > 0$, $x_2 > 0$). Analogously we shall write the following three rations of the spectra: $\alpha_2 = \tilde{J}(x_3)/\tilde{J}(x_2)$ in the third quadrant ($x_1, x_2 < 0$) and in the second quadrant ($x_1 < 0$, $x_2 > 0$), $\alpha_3 = \tilde{J}(x_4)/\tilde{J}(x_3)$ in the fourth quadrant ($x_1 > 0$, $x_2 < 0$) and the third quadrant ($x_1, x_2 < 0$), as well as $\alpha_4 = \tilde{J}(x_1)/\tilde{J}(x_4)$ in the first and fourth quadrants.

Assume that at the kth iteration the estimate of the Fourier spectrum $\tilde{J}_k(x)$ with the ratio β is obtained: $\beta_1 = \tilde{J}_k(x_2)/\tilde{J}_k(x_1)$, $\beta_2 = \tilde{J}_k(x_3)/\tilde{J}_k(x_2)$, $\beta_3 = \tilde{J}_k(x_4)/\tilde{J}_k(x_3)$, and $\beta_4 = \tilde{J}_k(x_1)/\tilde{J}_k(x_4)$. $\tilde{J}_k(x)$ must be corrected in the minimum manner so that the ratio β would equal α. From here it follows that the following functional must be minimized:

$$\sum_x |\tilde{I}_k(x_1) - \tilde{J}_k(x_1)|^2 + |\tilde{I}_k(x_2) - \tilde{J}_k(x_2)|^2 + |\tilde{I}_k(x_3) - \tilde{J}_k(x_3)|^2 + |\tilde{I}_k(x_4) - \tilde{J}_k(x_4)|^2 = \min.$$

where $\tilde{I}_k(x_1)$ is the new estimate sought for the spectrum in the ith quadrant with the true ratio α, while $\tilde{J}_k(x_1)$ is the estimate obtained for the spectrum in the ith quadrant with the ratio β. We shall write a more detailed expression for the functional:

$$\sum_{x} \sum_{i=1}^{4} \left\{\tilde{I}_k(x_1) - \tilde{J}_k(x_1)\right\}\left\{\tilde{I}^*_k(x_1) - \tilde{J}^*_k(x_1)\right\} = \min,$$

where ** denotes the complex conjugation. Differentiating this expression with respect to $\tilde{I}_k(x_1)$, equating the derivative to zero, using the antisymmetry of the spectrum $\tilde{I}_k(-x_1, -x_2) = \tilde{I}^*_k(x_1, x_2)$ and dropping intermediate calculations, we find the expression for the new spectrum in each quadrant $i = 1, 2, 3$, and 4:

$$I_k(x_1) = \frac{\tilde{J}_k(x_1)(1 + \alpha^*_1\beta_1) + J^*_k(x_1)\alpha^*_1\alpha^*_{1+1}(1 + \alpha_1\beta^*_1)}{2(1 + |\alpha_1|^2)}$$

The operator $\hat{P}_3$ in thus determined as $\hat{P}_3 J_k = \hat{F}^{-1}\{\tilde{I}_k\}$, where $\tilde{J}_k = \hat{F}\{J_k\}$, ** denotes complex conjugation, and α_1 and β_1 are, respectively, the ratio of the true spectrum and the Fourier spectrum obtained in the ith quadrant.

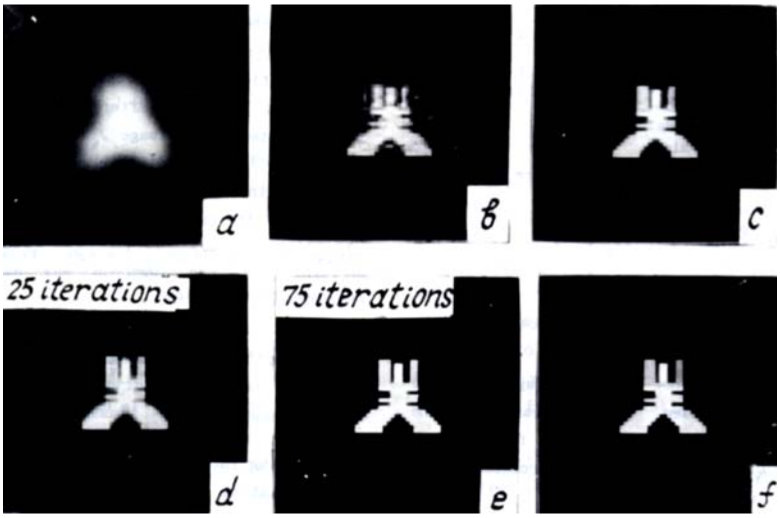

FIG. 1. a) Initial blurred image (the magnitude of the blurring is 9×9); b, c) results of reconstruction based on the relation (3) after 25 and 75 iterations, respectively; d, e) results of reconstruction based on the relation (5) after 25 *and* 75 *iterations, respectively; f) the true image.*

In conclusion we note the following:

a) The form of the defocusing function does not affect the formulation and solution of problem. Only the condition that the function be even or circular symmetry is important.

b) The distortions in the model calculations were given by multiplying the Fourier spectrum of the image by the Fourier spectrum of the pulse response of the blurring, equal to unity, within the square region $a \times a$ ($a = 9$). The transfer function in this case is equal to $\dfrac{\sin(ax_1 + ax_2)}{ax_1 + ax_2}$.

c) The algorithms developed were checked experimentally by modeling mathematically on a SM-1420 computer with 64 × 64 arrays. We assumed that the true dimensions of the image (length and width) were known and the action of the operator $\hat{P}_2$ reduced to multiplication by a rectangular mask. The typical results of the reconstruction are presented in Fig. 1. The modeling showed that the relative rms error of reconstruction not worse than 1% is achieved after 50–100 iterations (depending on the magnitude of the blurring).

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