Processes in a ring interferometer: the problem of describing by discrete maps

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Processes occurring in a nonlinear ring interferometer (NRI) are described using discrete maps in the case of a mono- and double-frequency optical field. The effect of physical factors on the stability of modes in the model processes proceeding in the NRI is investigated. Some techniques are proposed for interpolation and quantitative analysis of the morphology of the process stability maps in mathematical models.

1. Introduction. Discrete maps as a language for describing the dynamic systems

A four-mirror interferometer, including a nonlinear element, is called a nonlinear ring interferometer (NRI). As was demonstrated by Ikeda et al. (1979), the delay τ of the optical field as it passes in NRI is an important parameter of an actual interferometer. The further investigations by Akhmanov, Vorontsov, and Shmalgauzen with their colleagues from Moscow State University, as well as by Rozanov, and some researchers abroad have shown that the following regimes are possible to occur in the cross section of a laser beam in the NRI: autowaves, generation of static and moving structures, optical turbulence, intermittence, and chaos.[1] Therefore, the model of processes in the NRI is one of those models that cover numerous phenomena of nonlinear dynamics.

Figure 1 shows schematically the ring interferometer containing a thin layer of a nonlinear medium, whose refractive index depends on the amplitude of the electric field of the incident radiation (for example, as in the case of Kerr effect).

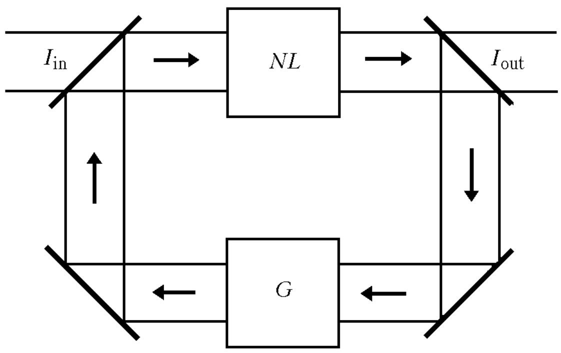

Fig. 1. Schematic diagram of a nonlinear ring interferometer with a two-dimensional feedback: NL is a nonlinear element; G is an element converting the field; $I_{\rm in}$ is the intensity of the input light field; $I_{\rm out}$ is the intensity of the output light field.

The main difference of this NRI from the Ikeda's model is that the feedback loop of the NRI includes a linear element G providing for point-by-point field conversion (shift, tilt, turn, expansion, compression). Thus, the ray trajectory starting at the point with the coordinates (x,y) ends at the point $(x', y') = G(x, y)$ upon round travel through the NRI.

As known, the behavior of dynamic systems can be described by evolution equations of the form

$$\tau_n {\rm d}q/{\rm d}t = N[q(t),\ q(t-\tau)], \tag{1}$$

where τ_n is the characteristic relaxation time.

For example, in the static mode, that is, with no changes in time (${\rm d}q/{\rm d}t = 0$), and if the equation $N[q(t), q(t-\tau)] = 0$ can be solved for $q(t)$, model (1) is reduced to the recurrence: $q(t) = F[q(t-\tau)]$. Then we can pass on to the model describing the state of a dynamic system at equidistant time moments $t_n = t_0 + \tau_n$:

$$q_n = F(q_{n-1}). \tag{2}$$

The equation of type (2), in the mathematical form, is an m-dimensional discrete map (DM), where m is the number of vector components.[2]

An advantage of the models based on DM is that they can be organically implemented on digital computers. From the viewpoint of mathematical simulation, the use of the DM apparatus (within its applicability domain) is advantageous over the finite-difference schemes employed in solving the Eq. (1). Operation with the DMs almost relieves the problems typical in solution of differential equations: approximation, stability, and errors of numerical methods. Owing to these advantages, the DM method has become popular in the investigations of model nonlinear dynamic systems.

In the most papers available in literature, the processes in the NRI are simulated for a particular case of single-frequency radiation. For the model to be more realistic, assume that double-frequency (bichromatic) radiation is entered into the NRI. This situation has been considered only in the most recent papers.[3,4]

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Within thus stated problem, it is reasonable to expect some new features in the complex dynamics of the processes in the model under the assumptions accepted. This, in turn, will extend our understanding of optical nonlinear ring systems. Besides, the application of the DM apparatus allows one to describe nontrivial phenomenon of deterministic spatial chaos.[5] This apparatus is also promising for nonlinear dynamic information safety systems as applied to data transfer (in the static mode) and storage.

The aim of this paper is to describe the NRI processes using the DM language for the cases of monochromatic and bichromatic optical fields in the approximation of high loss, as well as to study the effect of physical factors on the stability of modes in the corresponding model NRI processes.

To reveal the type of dynamics, let us have a look at the calculation of the Lyapunov characteristic exponents (LCE), which serves a quantitative measure of process instability. Having known the LCE, one can readily determine the fractal dimension of an attractor, entropy of a dynamic system, and the characteristic predictability time of the system behavior.[6]

2. Study of modes in model processes in the NRI based on discrete mapping

For identifying the stability of dynamic modes and revealing the effect of physical factors, the so-called LCE maps were drawn in the coordinates of model parameters.

2.1. Discrete mapping as a model of processes in the NRI: the case of monochromatic radiation

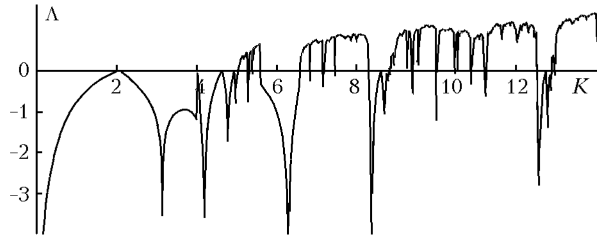
The considered model of the optical system with a two-dimensional feedback accounts for the dependence of the refractive index on the squared strength of the optical field and for the interference of light fields. Therefore, the nonlinear phase incursion U, rather than intensity is a representative dynamic variable. From the differential equation describing the dynamics of U (in the approximation of high loss, when the propagation of a light wave in the interferometer is considered in the geometric-optics approximation[3]) for the case of a single-frequency field constant both in time and in the cross section plane xOy, practicing the approach described in Sect. 1, one can obtain a discrete map of the following form:

$$\begin{cases} U_0 = \text{const} = K, \\ U_i = K\left[1 + \gamma\cos(U_{i-1} + \Phi)\right], \; i \in [1;\; \infty]. \end{cases} \quad (3)$$

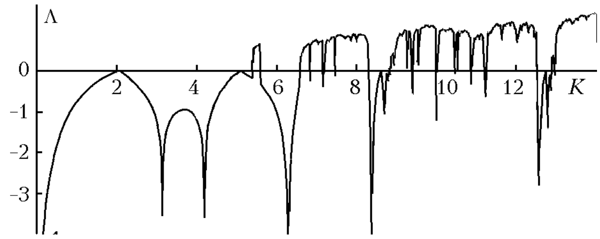
Here K is the coefficient of nonlinearity; γ is the doubled coefficient of radiation loss for an NRI path; Φ is the linear phase incursion in the interferometer feedback loop.

The results obtained by simulation (3) are shown in Figs. 2, 3, and 4.

The dependence $\Lambda(K)$ is nonmonotonic, and it experiences sign alternation or takes zero values (see Fig. 2). According to the theory, this behavior corresponds to the alternation of modes (bifurcations). As known, different bifurcations of the states occur just at $\Lambda = 0$ or $\mathrm{Re}(\tilde{\Lambda}) = 0$ (in the process of calculation of $\tilde{\Lambda}$ as Jacobian's eigenvalues, that is, the matrix of linearization).



a



b

Fig. 2. LCE dependence on the nonlinearity parameter K under different initial conditions: U_0 = 0.1 (*a*) and 2 (*b*).

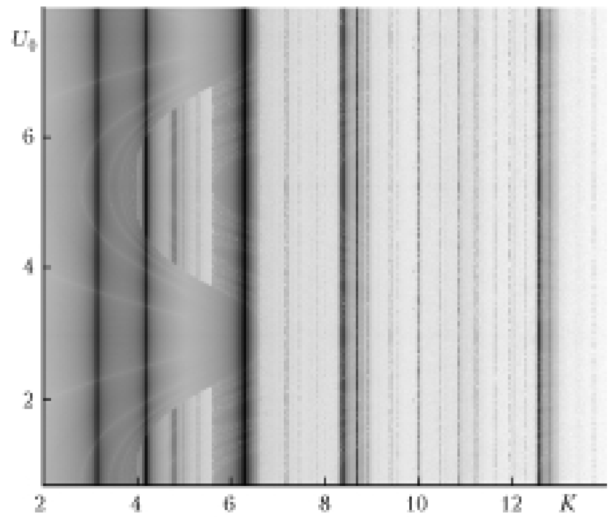


Fig. 3. LCE map in the coordinates $K - U_0$ (γ = 0.5).

Actually, from the comparison of the structure of bifurcation diagrams (BDs) (see Fig. 5) and the plot $\Lambda(K)$ in Fig. 2, it can be seen that the points with $\Lambda = 0$ correspond to the BD singular points, at which bifurcations occur. Thus, the correspondence of the points with $\Lambda = 0$ to the bifurcation points indicates that the obtained dependence is mathematically correct.

Note that the first equation in Eq. (3) is, essentially, the initial condition for this system. Besides, note that $\Lambda(K)$ depends on U_0. To reveal the effect of U_0, we have drawn the LCE map in the coordinates $K - U_0$ (Fig. 3). Actually, it reflects the dependence of the presence (absence) of stability of the steady-state mode on the initial state U_0.

As a further development of this approach, combine the LCE maps for DM with BDs of static states U for ordinary differential equations. This proves to be methodically a rather efficient approach, which facilitates morphological interpretation of maps and explains the presence of regular (light areas in Fig. 6) or chaotic (dark areas) dynamics in the model.

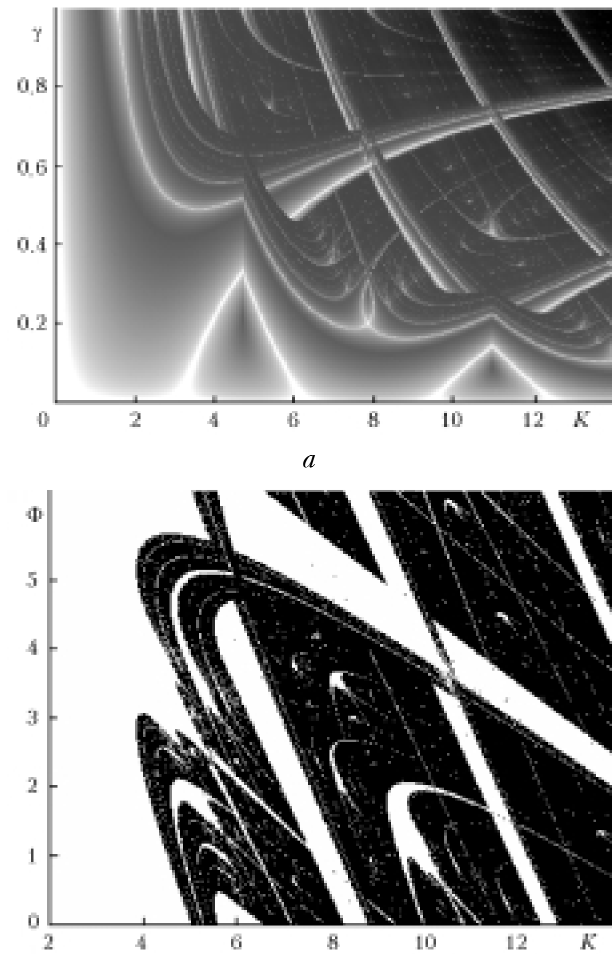


Fig. 4. LCE maps in coordinates $K - \gamma$ (*a*), $K - \Phi$ (*b*). Light areas correspond to negative Λ.

Thus, if the BDs include steady states (bold lines), then in the LCE map they correspond to the areas of the initial conditions, belonging to them, that result in a regular behavior of the system. The dependence $\Lambda(U_0)$ has the period 2π (Fig. 3). The correctness of the approach based on the joint analysis of the map and BDs assumes drawing as many as possible BDs for different number of ordinary differential equations.

As a further development of analogy of the descriptions that use ordinary differential equations (ODEs) and DMs, point out that the structure of the LCE map for DM (Fig. 4*b*) is conditioned by the structure of the family of bifurcation lines for the DE (Fig. 7). In Fig. 5, between the nearest pairs of lines *1* and *4*, *2* and *3*, there are stable parts of the branches (corresponding to the states $U_1 = U_2$) of the bifurcation diagrams, which are independent of the number of equations m. It can easily be seen that the structure of the maps copies the configuration of the areas between the bifurcation lines.

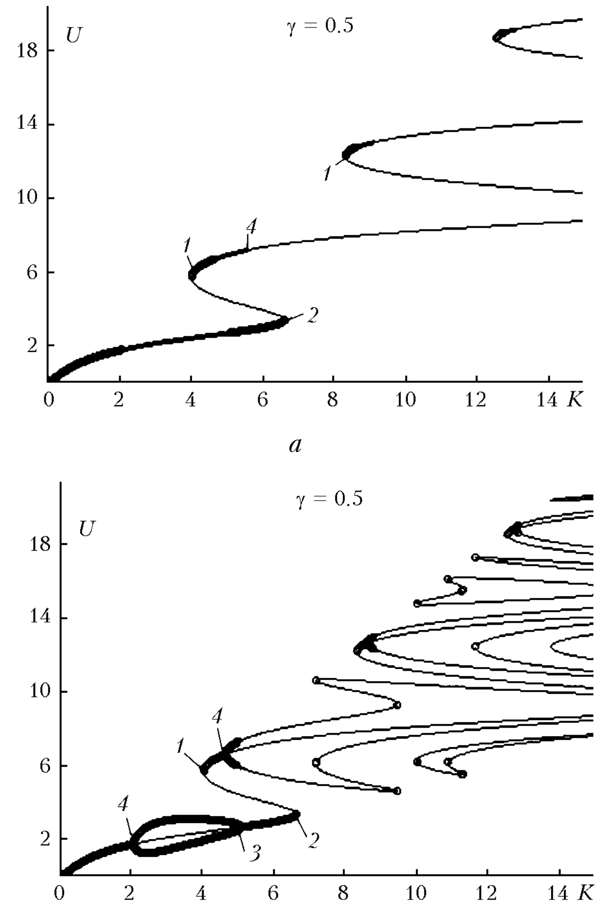


Fig. 5. Bifurcation diagrams of static states of the nonlinear phase incursion at $\tau = \tau_n$ for $m = 1$ (*a*) and 2 (*b*). Bold lines stand for the states steady at any τ, ordinary lines are for the states steady at given τ, and thin lines are for the unsteady states.

The found elements of similarity in the structure of the LCE maps for different situations, as well as the similarity with the results obtained earlier for the model based on ODEs, evidence the appropriateness of describing the processes in the NRI using DMs.

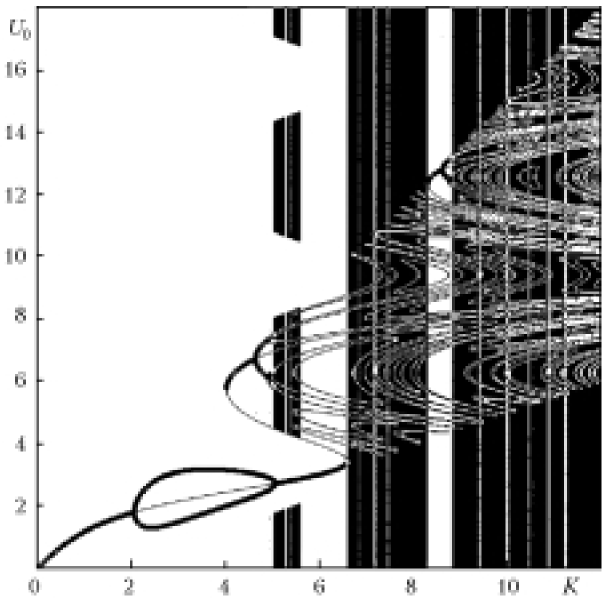


Fig. 6. Combination of "contrast" maps (sgn(λ)) on one plane with the bifurcation diagram (BD) of the static states U for four ordinary differential equations.

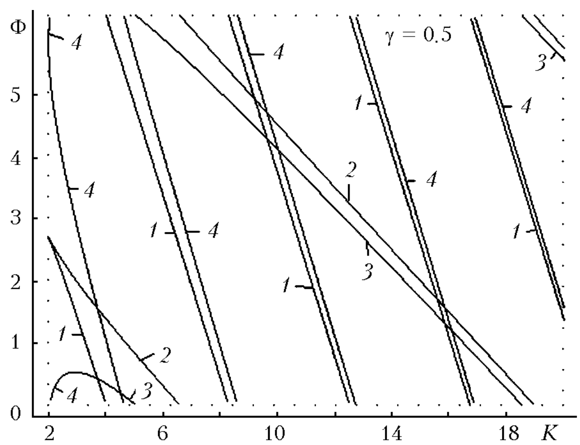


Fig. 7. Bifurcation lines $\Phi = \omega t_0$, that is, bifurcation parameter Φ as a function of the nonlinearity coefficient K. The value of Φ is normalized to the range [0; 2π], $m = 2$: discontinuous bifurcations of the appearance of a new steady-state solution (1) (*1*); discontinuous bifurcations of the disappearance of an old steady-state solution (1) (*2*); bifurcation of the acquiring stability (*3*); bifurcation of loosing stability (*4*).

2.2. The case of bichromatic radiation

Let the optical field at the NRI input consist of the components with the amplitudes a and b and the frequencies $\omega + \Omega$ and $\omega - \Omega$ (Fig. 8).

In accordance with the goal of this paper, consider now the double-frequency case, for which we can expect qualitative changes in the maps. Checking the correctness of this assumption and proceeding by analogy with derivation of Eq. (2), we can easily construct the DM for this case

$$U_{i+1} = K\,\{1 + \gamma\{Q_a \cos[(1+q)(\Phi + U_i)] + (1 - Q_a)\cos[(1-q)(\Phi + U_i)]\}\}, \qquad (4)$$

where Q_a is the fraction of intensity for the component with the frequency $(1 + q)\,\omega$; $q \equiv \Omega/\omega$ is the parameter of bichromaticity, which characterizes the interval between the spectral components (Fig. 8); Φ is the linear phase incursion at the frequency ω in the NRI feedback loop.

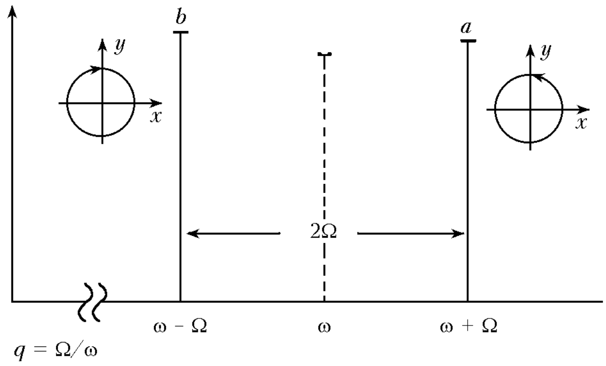


Fig. 8. Spectrum of bichromatic radiation.

Figure 9 gives an idea of the structure of the LCE maps in the plane $K - U_0$ at a relatively large (0.5) deviation of the parameter q from the value corresponding to the field monochromaticity.

Figure 10 demonstrates qualitative changes in the maps due to the appearance of the second component in the spectrum of the input optical field.

Because of specific differences in the description, comparison, and analysis of the morphology of maps, it is quite logical to look for some relatively objective quantitative characteristics of a map as a whole. In our opinion, for making a comparison of LCE maps with each other, the total area of regions, in the parameter space, corresponding to the positive LCE values (dark regions in the "contrast" maps) can serve such a characteristic.

The change in the ratio of the dark and light areas in the maps of $\Lambda(K, \gamma)$ while varying the parameters q, Q_a is depicted in Fig. 11. Each of these dependences is nonmonotonic and includes a pronounced extreme near $q = 0.16$ (Fig. 11*a*) and $Q_a = 0.34$ (Fig. 11*b*).

A more systematic morphological interpretation of these tendencies calls for an independent study and is beyond the scope of this paper.

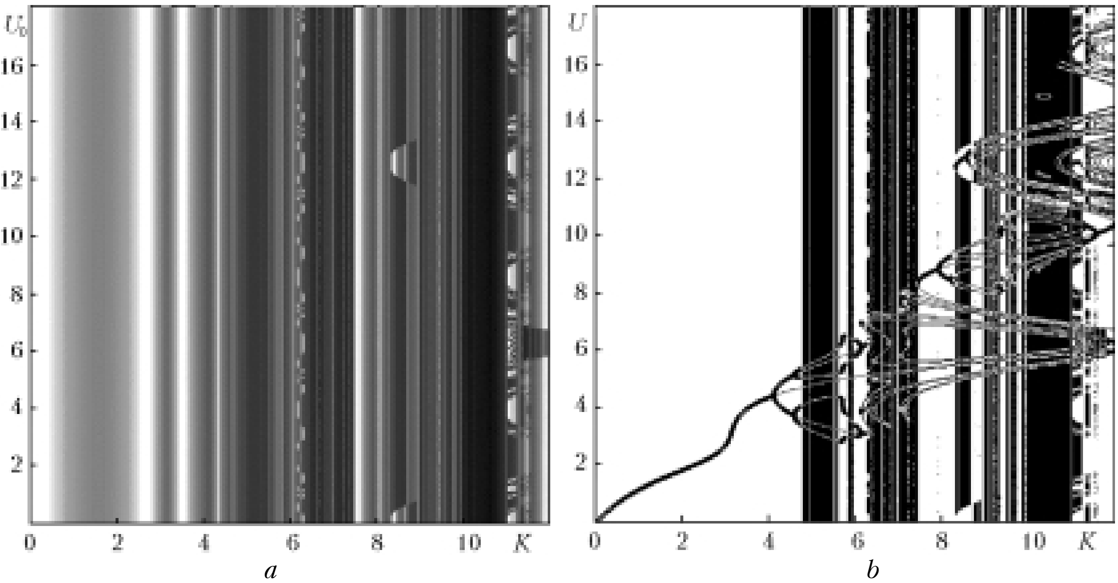


Fig. 9. Bichromatic case, LCE map in the coordinates $K - U_0$ (a), combination of the "contrast" map ($\mathrm{sgn}(\lambda)$) with the BD of static states (b).

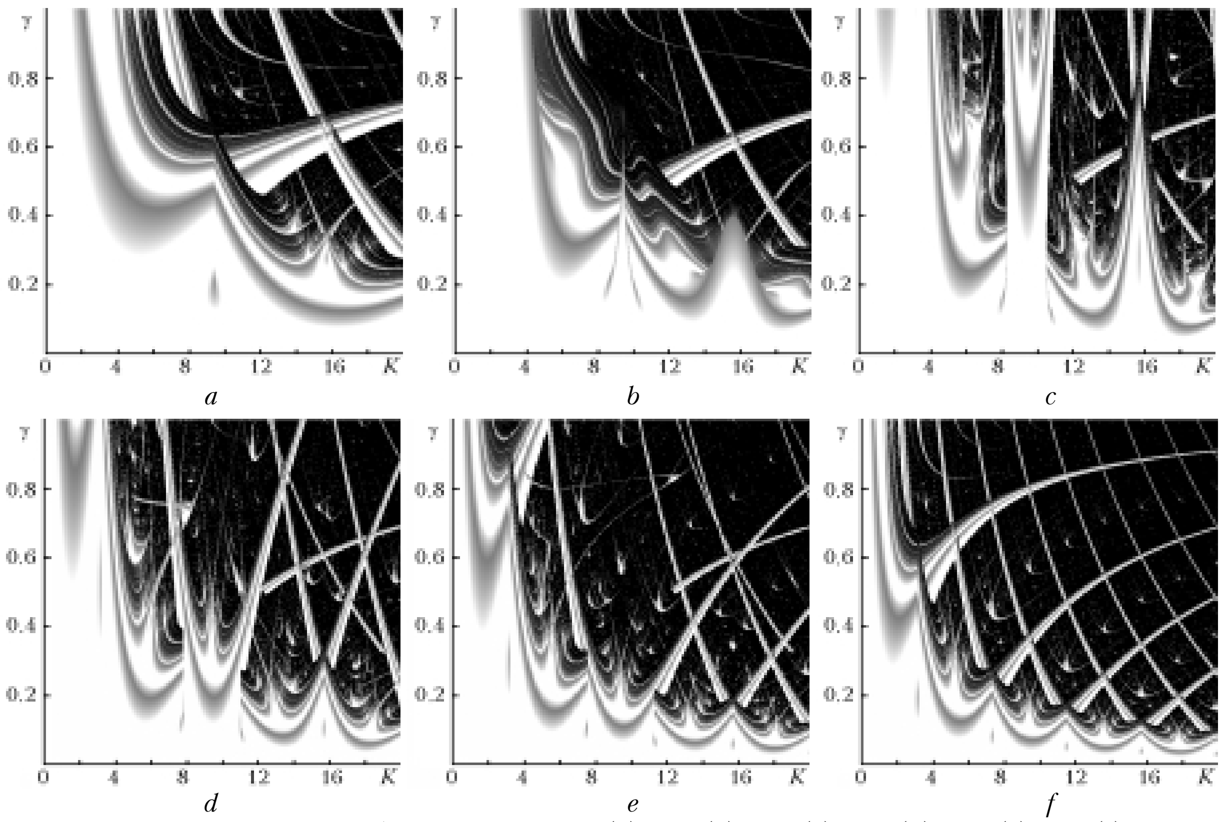


Fig. 10. LCE maps in the coordinates $K - \gamma$, Q_a = 0 (a), 0.17 (b), 0.35 (c), 0.52 (d), 0.73 (e), 0.99 (f).

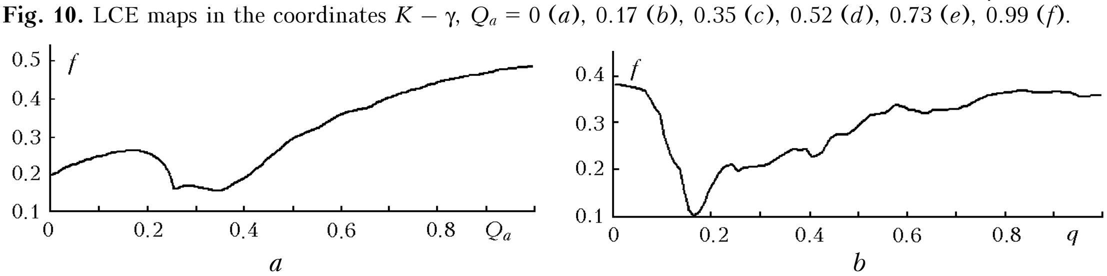


Fig. 11. Total area of the regions corresponding to positive LCE values (for the maps in the coordinates $K - \gamma$).

Conclusion

The processes proceeding in the NRI have been described in the DM language for the cases of single- and double-frequency field in the approximation of high loss. The effect of physical factors on the stability of modes in the corresponding model processes proceeding in the NRI has been studied. For this purpose, we have drawn a set of LCE maps, that is, the distributions of the LCE values in the planes of pairs of the model parameters and in the plane "nonlinearity parameter – the initial condition."

It has been found that the structure of the maps change qualitatively with the second spectral component introduced in the radiation at the NRI input.

The technique has been proposed for interpretation of the LCE maps drawn. This technique is uses BDs and bifurcation lines obtained for the model using ordinary differential equations. Their mutual conditionality was revealed, which indicates the appropriateness of using the discrete maps.

To facilitate the morphological analysis of the maps, it has been proposed to apply to the maps the BDs constructed for the ODEs.

For quantitative characterization of the LCE map, it has been proposed to use its fraction corresponding to the unsteady-state mode.

The results obtained are of interest for atmospheric adaptive optics, whose instrumentation includes NRIs.[7]

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