A new method to allow for the refraction in radiative transfer equation within the model of spherical atmosphere

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A new method is proposed to allow for the refraction in the equation of radiation transfer through the atmosphere. The method is based on deformation of the spherical coordinate system in accordance with the spatial distribution of the refractive index. It is shown that in the deformed coordinate system the differential operator of the transfer equation that allows for the refraction takes a simpler form corresponding to the rectilinear propagation of light and contains no refraction terms. The result obtained simplifies formulation of the problems of the radiation transfer theory for a refractive spherical atmosphere.

Introduction

To study the scattered solar radiation, atmospheric optics usually uses the model of the planetary atmosphere as a plane layer illuminated by parallel rays and limited by a reflecting surface from the bottom. However, in many problems, radiation scattering in the atmosphere should be considered with the allowance for the atmospheric sphericity and the effect of ray bending due to refraction. At rigorous consideration of the refraction in a spherical atmosphere, calculated results may significantly differ from that obtained assuming rectilinear propagation of light.[1]

The theory of refraction[2] is based on the differential equation of refraction describing the change in the direction of a ray propagating in a medium with a variable refractive index. Therefore, the radiative transfer equation (RTE) becomes more complicated when considering light propagation in a spherical atmosphere.[1] Thus, a question naturally arises on whether or not the RTE can be reduced to a simpler form corresponding to the rectilinear propagation of rays. In this paper, we give an affirmative answer to this question and justify this possibility.

Analysis of RTE structure when considering refraction in a spherical atmosphere

In a spherical coordinate system with the axis *OZ* directed toward local zenith, the radiative transfer equation accounting for the refraction can be presented in the following form[1]:

$$\cos\vartheta\frac{\partial}{\partial r}\left(\frac{I}{n^2}\right)+\frac{\sin\vartheta\cos\varphi}{r}\frac{\partial}{\partial\psi}\left(\frac{I}{n^2}\right)-$$

$$-\frac{\sin\vartheta}{r}\left(1-\frac{r}{r_\mathrm{c}}\right)\frac{\partial}{\partial\vartheta}\left(\frac{I}{n^2}\right)-\frac{\sin\vartheta\sin\varphi\ \mathrm{cotan}\psi}{r}\frac{\partial}{\partial\varphi}\left(\frac{I}{n^2}\right)=$$

$$=\varepsilon\left(\frac{B}{n^2}-\frac{I}{n^2}\right). \qquad (1)$$

It differs from the corresponding equation on the assumption of rectilinear light propagation[3] by the refraction factor $(1\ \$\ r/r_\mathrm{c})$, the function B/n^2, and the function I/n^2 corresponding to variation of the intensity I along the ray in the medium with the variable refractive index $n = n(\mathbf{r})$. The value of I/n^2 follows from the law of conservation for the light beam as a ray tube.[4] In a homogeneous medium at n = const, the invariant I/n^2 turns into I. At the solar illumination under conditions of the axial symmetry, the intensity $I(r, \psi, \vartheta, \varphi)$ is a function of coordinates of the point $\mathbf{r}(r, \psi)$ and the direction $\mathbf{s}(\vartheta, \varphi)$, where r and ψ are, respectively, the radius and the polar angle determining the point's position; ϑ and φ are the polar and azimuth angles of the vector of beam direction. In Eq. (1) B is the source function; ε is the extinction coefficient of the medium; the refraction caused curvature of the ray $1/r_\mathrm{c}$ (r_c is the length of curvature) is expressed through the logarithmic derivative of the vertical profile of the refractive index with respect to r:

$$\frac{1}{r_\mathrm{c}}=-\frac{\partial}{\partial r}\ln n\,. \qquad (2)$$

The deviation of the factor $(1\ \$\ r/r_\mathrm{c})$ from unity characterizes the effect of refraction. In the surface atmospheric layer, the refraction effect is marked: $(1\ \$\ r/r_\mathrm{c}) = 0.77$ (Ref. 1).

Let us derive the equation for the differential operator of RTE (1), analyze its structure, and find what parameters determine every its component. As known, the differential RTE operator can be written in the coordinateless form as a scalar product $(\mathbf{s}, \nabla I)$ of the vector of ray direction and the intensity gradient. To specify the point's position $P(\mathbf{r})$ in a spherical atmosphere, the spherical system of coordinates (r, ψ, ϕ) with the origin at the center of a planet, where ϕ is the azimuth angle, is used. The spherical system of coordinates is considered as a particular case of a curvilinear system $\{u_k,\ k = 1, 2, 3\}$: $u_1 = r$, $u_2 = \psi$, $u_3 = \phi$. The direction vector **s** is described by the spherical coordinates $(\rho, \vartheta, \varphi)$. At the first stage of transformations, we assume that the vector length $\rho \neq 1$.

The gradient ∇I of the intensity field $I = I(\mathbf{r}, \mathbf{s}) = I(r, \psi, \phi, \vartheta, \varphi, \rho)$ in the (generalized) six-dimensional phase space can be written as[5]:

$$\nabla I = \sum_{k=1}^{3} \frac{\mathbf{e}_k}{H_k} \left(\frac{\partial I}{\partial u_k} + \frac{\partial I}{\partial \vartheta}\frac{\partial \vartheta}{\partial u_k} + \frac{\partial I}{\partial \varphi}\frac{\partial \varphi}{\partial u_k} + \frac{\partial I}{\partial \rho}\frac{\partial \rho}{\partial u_k} \right), \qquad (3)$$

where H_k are the Lame coefficients determining the curvature of the coordinate lines u_k; $\mathbf{e}_k$ is the basis of orthogonal unit vectors. The Lame coefficients for the spherical system of coordinates are[6]:

$$H_r = 1,\ H_\psi = r,\ H_\phi = r \sin\psi. \qquad (4)$$

The peculiarity of the spherical coordinate system complicating the consideration is that, unlike the Cartesian coordinate system, the orientation of the basis of unit vectors $\{\mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_\phi\}$ in the spherical system depends on the point's position P in space. Let us show that at every point P the transition $\{\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z\} \to \{\mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_\phi\}$ from the basis of unit vectors of the Cartesian coordinate system to the basis corresponding to the spherical coordinates with the local zenith can be made by applying the transformation $\hat{A} = \hat{M}\,\hat{G}$. The matrix $\hat{M}$ can be expressed through the Lame coefficients H_r, H_ψ, H_ϕ and partial derivatives of the Cartesian coordinates $X = r \sin\psi \cos\phi$, $Y = r \sin\psi \sin\phi$, $Z = r \cos\psi$ as

$$\hat{M} = \begin{bmatrix} \frac{1}{H_r}\frac{\partial X}{\partial r} & \frac{1}{H_r}\frac{\partial Y}{\partial r} & \frac{1}{H_r}\frac{\partial Z}{\partial r} \\ \frac{1}{H_\psi}\frac{\partial X}{\partial \psi} & \frac{1}{H_\psi}\frac{\partial Y}{\partial \psi} & \frac{1}{H_\psi}\frac{\partial Z}{\partial \psi} \\ \frac{1}{H_\phi}\frac{\partial X}{\partial \phi} & \frac{1}{H_\phi}\frac{\partial Y}{\partial \phi} & \frac{1}{H_\phi}\frac{\partial Z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\psi\cos\phi & \sin\psi\sin\phi & \cos\psi \\ \cos\psi\cos\phi & \cos\psi\sin\phi & -\sin\psi \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}. \qquad (5)$$

The matrix $\hat{G}$ describes the orthogonal transformation $\hat{G}(\phi, \psi) = \hat{T}_Z(\phi)\,\hat{T}_Y(\psi)$ corresponding to turns about the axes OZ and OY through the angles ϕ and ψ (Ref. 7):

$$\hat{G}(\phi, \psi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\phi & -\sin\phi & \sin\psi\cos\phi \\ \cos\psi\sin\phi & \cos\phi & \sin\psi\sin\phi \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}. \qquad (6)$$

As a result, we have an orthogonal matrix

$$\hat{A} = \hat{M}\,\hat{G} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad (7)$$

which allows us to find the coordinates a_k of the direction vector $\mathbf{s} = \sum_{k=1}^{3} a_k \mathbf{e}_k$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_\phi\}$ of spherical coordinates. Actually, the spherical coordinates a_k are linearly related to the Cartesian coordinates $x = \rho \sin\vartheta \cos\varphi$, $y = \rho \sin\vartheta \cos\varphi$, $z = \rho \cos\vartheta$ through the transformation matrix $\hat{A}$ (7):

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \rho\sin\vartheta\cos\varphi \\ \rho\sin\vartheta\sin\varphi \\ \rho\cos\vartheta \end{bmatrix} = \begin{bmatrix} \rho\cos\vartheta \\ \rho\sin\vartheta\cos\varphi \\ \rho\sin\vartheta\sin\varphi \end{bmatrix}. \qquad (8)$$

After scalar multiplication of the vector **s** (8) by the vector ∇I (3), we find the equation for the differential RTE operator in the curvilinear coordinates:

$$(\mathbf{s}, \nabla I) = \sum_{k=1}^{3} a_k \frac{1}{H_k}\frac{\partial I}{\partial u_k} + \left(\sum_{k=1}^{3} a_k \frac{1}{H_k}\frac{\partial \vartheta}{\partial u_k} \right)\frac{\partial I}{\partial \vartheta} + \left(\sum_{k=1}^{3} a_k \frac{1}{H_k}\frac{\partial \varphi}{\partial u_k} \right)\frac{\partial I}{\partial \varphi} + \left(\sum_{k=1}^{3} a_k \frac{1}{H_k}\frac{\partial \rho}{\partial u_k} \right)\frac{\partial I}{\partial \rho}. \qquad (9)$$

Equation (9) includes unknown partial derivatives $\frac{\partial \vartheta}{\partial u_k}, \frac{\partial \varphi}{\partial u_k}, \frac{\partial \rho}{\partial u_k}$ of the direction vector with respect to the curvilinear coordinates.

Let us analyze now the structure of fields formed by the partial derivatives. To find them, we used the results of Ref. 5. Take into account that the differentiation operator in the curvilinear coordinates acts not only on the components a_i of the vector, but also on the basis vectors $\mathbf{e}_i$:

$$\mathbf{o}_k = \frac{\partial \mathbf{s}}{\partial u_k} = \sum_{i=1}^{3} \left(\frac{\partial a_i}{\partial u_k}\mathbf{e}_i + \frac{\partial \mathbf{e}_i}{\partial u_k} a_i \right), \quad i = 1, 2, 3. \qquad (10)$$

Substitution of the well known equation[5]

$$\frac{\partial \mathbf{e}_i}{\partial u_k} = (1 - \delta_{ik})\frac{1}{H_i}\frac{\partial H_k}{\partial u_i}\mathbf{e}_k - \delta_{ik} \sum_{s=1}^{3} (1 - \delta_{sk})\frac{1}{H_s}\frac{\partial H_k}{\partial u_s}\mathbf{e}_s, \quad i = 1, 2, 3, \qquad (11)$$

into Eq. (10) and calculation of the scalar products $(\mathbf{o}_k, \mathbf{e}_j)$, $j = 1, 2, 3$, with the allowance for the fact that the basis is orthonormal $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, yields the equation

$$\frac{\partial \mathbf{a}}{\partial u_k} = \hat{Q}_k\,\mathbf{a} + \mathbf{f}_k = \hat{Q}_k\,\hat{A}\,\mathbf{x} + \mathbf{f}_k. \qquad (12)$$

Here $\hat{Q}_k$ are antisymmetric matrices; $\mathbf{a} = \hat{A}\,\mathbf{x}$; $\mathbf{a} = (a_r, a_\psi, a_\phi)^{\mathrm{T}}$, $\mathbf{x} = (x, y, z)^{\mathrm{T}}$, $\mathbf{f}_k = (\mathbf{o}_k\mathbf{e}_1,\ \mathbf{o}_k\mathbf{e}_2,\ \mathbf{o}_k\mathbf{e}_3)^{\mathrm{T}}$ are algebraic vectors, and T denotes transposition.

Introducing the designations

$$\hat{C} = \begin{bmatrix} \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \rho} \\ \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \rho} \\ \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \rho} \end{bmatrix},$$

$$\hat{B} \equiv \mathbf{b}_k = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial \vartheta}{\partial r} & \frac{\partial \vartheta}{\partial \psi} & \frac{\partial \vartheta}{\partial \phi} \\ \frac{\partial \varphi}{\partial r} & \frac{\partial \varphi}{\partial \psi} & \frac{\partial \varphi}{\partial \phi} \\ \frac{\partial \rho}{\partial r} & \frac{\partial \rho}{\partial \psi} & \frac{\partial \rho}{\partial \phi} \end{bmatrix}, \quad k = 1, 2, 3, \quad (13)$$

meeting the obvious equality

$$\frac{\partial \mathbf{x}}{\partial u_k} = \hat{C}\,\mathbf{b}_k, \quad (14)$$

with the allowance for Eq. (14), we obtain from Eq. (8) that

$$\frac{\partial \mathbf{a}}{\partial u_k} = \hat{A}\,\frac{\partial \mathbf{x}}{\partial u_k} = \hat{A}\,\hat{C}\,\mathbf{b}_k. \quad (15)$$

Equating the right-hand sides of Eqs. (12) and (15) and multiplying the resulting equality first by $\hat{A}^{-1}$ and then by $\hat{C}^{-1}$, we find the equation for the sought partial derivatives b_{ik}:

$$\mathbf{b}_k = \hat{C}^{-1}\,\hat{W}_k\,\mathbf{x} + \mathbf{F}_k. \quad (16)$$

Here we use the following designations:

$$\hat{W}_k = \hat{A}^{-1}\,\hat{Q}_k\,\hat{A}; \quad \mathbf{F}_k = \hat{C}^{-1}\,\hat{A}^{-1}\,\mathbf{f}_k. \quad (17)$$

The matrices $\hat{C}^{-1}$, $\hat{W}_k$, $k \in \{r, \psi, \phi\}$ are calculated by the following equations:

$$\hat{C}^{-1} = \frac{[A_{ik}]^{\mathrm{T}}}{D} = \begin{bmatrix} \frac{1}{\rho}\cos\vartheta\cos\varphi & \frac{1}{\rho}\cos\vartheta\sin\varphi & -\frac{1}{\rho}\sin\vartheta \\ -\frac{1}{\rho}\frac{\sin\varphi}{\sin\vartheta} & \frac{1}{\rho}\frac{\cos\varphi}{\sin\vartheta} & 0 \\ \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta \end{bmatrix}; \quad (18)$$

$$\hat{W}_r = 0,$$

$$\hat{W}_\psi = \begin{bmatrix} 0 & -\frac{1}{H_\phi}\frac{\partial H_\psi}{\partial \phi} & -\frac{1}{H_r}\frac{\partial H_\psi}{\partial r} \\ \frac{1}{H_\phi}\frac{\partial H_\psi}{\partial \phi} & 0 & 0 \\ \frac{1}{H_r}\frac{\partial H_\psi}{\partial r} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (19)$$

$$\hat{W}_\phi = \begin{bmatrix} 0 & \frac{1}{H_\psi}\frac{\partial H_\phi}{\partial \psi} & 0 \\ -\frac{1}{H_\psi}\frac{\partial H_\phi}{\partial \psi} & 0 & -\frac{1}{H_r}\frac{\partial H_\phi}{\partial r} \\ 0 & \frac{1}{H_r}\frac{\partial H_\phi}{\partial r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & 0 \\ -\cos\psi & 0 & -\sin\psi \\ 0 & \sin\psi & 0 \end{bmatrix}, \quad (20)$$

where A_{ik} is the algebraic adjunct of the element c_{ik}; $D = \rho^2 \sin\vartheta$ is the determinant of the matrix $\hat{C}$. Substitution of Eqs. (18)–(20) into Eq. (16) gives the explicit equations for the elements b_{ik} of the matrix $\hat{B}$:

$$\hat{B} = \begin{bmatrix} 0 & -\cos\varphi & -\sin\varphi\sin\psi \\ 0 & \mathrm{cotan}\vartheta\sin\varphi & -\cos\psi - \mathrm{cotan}\vartheta\cos\varphi\sin\psi \\ 0 & 0 & 0 \end{bmatrix}. \quad (21)$$

Refraction was ignored at derivation of Eq. (21), since it was assumed that the vector $\mathbf{F}_k = 0$ in Eq. (16). Besides, it was assumed that $\rho = 1$.

Substituting Eqs. (4), (8), and (21) into Eq. (9), reveals its inner structure and enables one to derive explicit equations for the terms of the differential RTE operator in the spherical coordinates:

$$(\mathbf{s}, \nabla I)|_r = a_r \frac{1}{H_r}\frac{\partial I}{\partial r} = \cos\vartheta \frac{\partial I}{\partial r}, \quad (22)$$

$$(\mathbf{s}, \nabla I)|_\psi = a_\psi \frac{1}{H_\psi}\frac{\partial I}{\partial \psi} = \frac{\sin\vartheta\cos\varphi}{r}\frac{\partial I}{\partial \psi}, \quad (23)$$

$$(\mathbf{s}, \nabla I)|_\phi = a_\phi \frac{1}{H_\phi}\frac{\partial I}{\partial \phi} = \frac{\sin\vartheta\sin\varphi}{r\sin\psi}\frac{\partial I}{\partial \phi}, \quad (24)$$

$$(\mathbf{s}, \nabla I)|_\vartheta = \left(a_r \frac{1}{H_r}\frac{\partial \vartheta}{\partial r} + a_\psi \frac{1}{H_\psi}\frac{\partial \vartheta}{\partial \psi} + a_\phi \frac{1}{H_\phi}\frac{\partial \vartheta}{\partial \phi}\right) \times \frac{\partial I}{\partial \vartheta} = -\frac{\sin\vartheta}{r}\frac{\partial I}{\partial \vartheta}, \quad (25)$$

$$(\mathbf{s}, \nabla I)|_\varphi = \left(a_r \frac{1}{H_r}\frac{\partial \varphi}{\partial r} + a_\psi \frac{1}{H_\psi}\frac{\partial \varphi}{\partial \psi} + a_\phi \frac{1}{H_\phi}\frac{\partial \varphi}{\partial \phi}\right) \times \frac{\partial I}{\partial \varphi} = -\frac{\sin\vartheta\sin\varphi\,\mathrm{cotan}\psi}{r}\frac{\partial I}{\partial \varphi}, \quad (26)$$

$$(\mathbf{s}, \nabla I)|_\rho = \left(a_r \frac{1}{H_r}\frac{\partial \rho}{\partial r} + a_\psi \frac{1}{H_\psi}\frac{\partial \rho}{\partial \psi} + a_\phi \frac{1}{H_\phi}\frac{\partial \rho}{\partial \phi}\right)\frac{\partial I}{\partial \rho} = 0. \quad (27)$$

Thus, the sum of nonzero components (22)–(26) gives the equation for the differential RTE operator in the five-dimensional phase space:

$$(\mathbf{s}, \nabla I) = \sum_{n=1}^{5} (\mathbf{s}, \nabla I)|_n, \quad n \in \{r, \psi, \phi, \vartheta, \varphi\}. \quad (28)$$

Under conditions of the spherical symmetry of the exposure to solar radiation, when $\frac{\partial I}{\partial \phi} = 0$, in Eq. (28) the term $(\mathbf{s}, \nabla I)\big|_{\phi} = 0$. In the case of the central symmetry $(\mathbf{s}, \nabla I)\big|_{\psi} = 0$, $(\mathbf{s}, \nabla I)\big|_{\phi} = 0$, $(\mathbf{s}, \nabla I)\big|_{\varphi} = 0$. For the model of a plane atmosphere, when $r = \infty$, the elements $(\mathbf{s}, \nabla I)\big|_{\psi} = 0$, $(\mathbf{s}, \nabla I)\big|_{\phi} = 0$, $(\mathbf{s}, \nabla I)\big|_{\vartheta} = 0$, $(\mathbf{s}, \nabla I)\big|_{\varphi} = 0$, and the differential operator contains only one term $(\mathbf{s}, \nabla I)\big|_{r} \neq 0$.

We can see that the term $(\mathbf{s}, \nabla I)\big|_{\vartheta}$ (25) is determined by the elements $b_{12} = \frac{\partial \vartheta}{\partial \psi}$ and $b_{13} = \frac{\partial \vartheta}{\partial \phi}$ of the matrix $\hat{B}$ (21), and the term $(\mathbf{s}, \nabla I)\big|_{\varphi}$ (26) is correspondingly determined by the elements $b_{22} = \frac{\partial \varphi}{\partial \psi}$ and $b_{23} = \frac{\partial \varphi}{\partial \phi}$. It is obvious that these nonzero elements of the matrix $\hat{B}$ express geometric properties of the spherical coordinate system used.

Consider how the elements b_{ik} of the matrix $\hat{B}$ change if we do not accept the condition $\mathbf{F}_k = 0$ of the rectilinear light propagation in Eq. (16). Actually, the vector $\mathbf{F}_k \neq 0$ if one accounts for the atmospheric refraction. Direct derivation of the components of the algebraic vector

$$\mathbf{F}_k = \hat{C}^{\$1} \hat{A}^{\$1} \left(\frac{\partial \mathbf{s}}{\partial u_k}, \mathbf{e}_j \right), \quad j = 1, 2, 3, \qquad (29)$$

on the assumption of the spherically symmetric distribution of the refractive index in space and substitution of the obtained equations into Eq. (16) lead to the nonzero complementary element of the matrix $\hat{B}$

$$b_{11} = \frac{\partial \vartheta}{\partial r} = \$ \tan \vartheta \frac{\partial}{\partial r} \ln n = \tan \vartheta \frac{1}{r_c}, \qquad (30)$$

which includes the parameter $\frac{1}{r_c}$ of the refraction curvature of a light ray. It can be easily seen that the term $(\mathbf{s}, \nabla I)\big|_{\vartheta}$ (25) at substitution into Eq. (30) takes the form

$$(\mathbf{s}, \nabla I)\big|_{\vartheta} = \$ \frac{\sin\vartheta}{r} \left(1 \$ \frac{r}{r_c} \right) \frac{\partial I}{\partial \vartheta} \qquad (31)$$

corresponding to the RTE component (1) responsible for refraction. Thus, from the above analysis we can see that the element b_{11} of the matrix $\hat{B}$ corresponds to the light refraction as a physical effect. This conclusion is important for further consideration.

New method to allow for refraction in RTE

When justifying the new method to allow for the refraction, we remain within the assumption $b_{11} = 0$, which determines the rectilinear trajectory of rays. Let us demonstrate that refraction in this case can be taken into account through some change of the elements b_{12}, b_{13}, b_{22}, and b_{23} that determine the geometric properties of the used coordinate system through, speaking figuratively, its deformation. Toward this end, let us change at every point P of the space the curvature of coordinate lines in accordance with the values of the refractive index. Besides the geometric radius r, below we also use the optical radius $r^{o} = nr$. The Lame coefficients expressing the curvature of the new coordinate lines can be represented in the following form:

$$H_r^{o} = n, \quad H_\psi^{o} = nr, \quad H_\phi^{o} = nr \sin\psi. \qquad (32)$$

With the allowance for Eq. (32), the nonzero antisymmetric matrices $\hat{W}_\psi^{o}$ and $\hat{W}_\phi^{o}$ that determine the values of the new elements b_{ik}^{o} (16) take the form

$$\hat{W}_\psi^{o} = \begin{bmatrix} 0 & 0 & -\left(1 - \frac{r}{r_c}\right) \\ 0 & 0 & 0 \\ \left(1 - \frac{r}{r_c}\right) & 0 & 0 \end{bmatrix},$$

$$\hat{W}_\phi^{o} = \begin{bmatrix} 0 & \cos\psi & 0 \\ -\cos\psi & 0 & -\sin\psi\left(1 - \frac{r}{r_c}\right) \\ 0 & \sin\psi\left(1 - \frac{r}{r_c}\right) & 0 \end{bmatrix} \qquad (33)$$

different than that of Eqs. (19) and (20) by the factors $\left(1 \$ \frac{r}{r_c}\right)$. In the following mathematical transformations according to Eq. (16), these factors transit into the equations for the elements of the matrix $\hat{B}^{o}$:

$$\hat{B}^{o} = \begin{bmatrix} 0 & -\cos\varphi\left(1 - \frac{r}{r_c}\right) & -\sin\varphi \sin\psi\left(1 - \frac{r}{r_c}\right) \\ 0 & \mathrm{cotan}\vartheta \sin\varphi\left(1 - \frac{r}{r_c}\right) & -\cos\psi - \mathrm{cotan}\vartheta \cos\varphi \sin\psi\left(1 - \frac{r}{r_c}\right) \\ 0 & 0 & 0 \end{bmatrix}. \qquad (34)$$

We can see from Eq. (34) that the refraction factors turn out to be hidden in the elements b_{ik}^{o} characterizing just the geometrical properties of the new deformed coordinate system. Substitute the elements of the matrix (34) into Eqs. (25) and (26)

$$(\mathbf{s},\nabla I)|_{\vartheta}^{o}=$$

$$=\left[0-\sin\vartheta\cos^2\varphi\frac{1}{r}\left(1-\frac{r}{r_c}\right)-\sin\vartheta\sin^2\varphi\frac{1}{r}\left(1-\frac{r}{r_c}\right)\right]\frac{\partial I}{\partial\vartheta}=$$

$$=-\frac{\sin\vartheta}{r}\left(1-\frac{r}{r_c}\right)\frac{\partial I}{\partial\vartheta}, \qquad (35)$$

$$(\mathrm{s},\nabla I)|_{\varphi}^{o}=\left[0+\cos\vartheta\sin\varphi\cos\varphi\frac{1}{r}\left(1-\frac{r}{r_c}\right)-\right.$$

$$\left.-\sin\vartheta\sin\varphi\,\mathrm{cotan}\psi\frac{1}{r}-\cos\vartheta\sin\varphi\cos\varphi\frac{1}{r}\left(1-\frac{r}{r_c}\right)\right]\frac{\partial I}{\partial\varphi}=$$

$$=-\frac{\sin\vartheta\sin\varphi\,\mathrm{cotan}\psi}{r}\frac{\partial I}{\partial\varphi} \qquad (36)$$

and make sure that Eqs. (35) and (36) obtained in the deformed spherical coordinates under the conditions of rectilinear light propagation coincide with the corresponding components of the differential RTE operator (1) that account for ray bending due to refraction. Replacement of the function I in the spherical coordinate system $\{r, \psi, \vartheta, \varphi\}$ with the function $I^o = I / n^2$ in the deformed coordinate system $\{r^o, \psi, \vartheta, \varphi\}$ with the Lame coefficients (32) reduces the RTE (1) to the form[3]:

$$\cos\vartheta\frac{\partial I^o}{\partial r^o}+\frac{\sin\vartheta\cos\varphi}{r^o}\frac{\partial I^o}{\partial\psi}-\frac{\sin\vartheta}{r^o}\frac{\partial I^o}{\partial\vartheta}-$$
$$-\frac{\sin\vartheta\sin\varphi\,\mathrm{cotan}\psi}{r^o}\frac{\partial I^o}{\partial\varphi}=\varepsilon\left(B^o-I^o\right), \qquad (37)$$

free of refraction. The radiative transfer equation in the spherical coordinates with the allowance for the refraction (1) is equivalent to the transfer equation in the deformed coordinate system (37), which does not include explicitly any refraction terms.

Conclusion

In this paper, we derived the equation for the differential operator of the radiative transfer equation in spherical coordinates with local zenith for the five-dimensional phase space in the model of spherical atmosphere with the refraction taken into consideration. This equation allows the inner structure of the RTE terms to be revealed from the positions of the field theory in curvilinear coordinates.

Vector analysis of the field of ray directions in the medium with the variable refractive index showed what elements of the matrix of partial derivatives determine, on the one hand, geometric characteristics of the coordinate system used and, on the other hand, parameters of the actual refraction curvature of the light ray.

The new method for considering refraction in the radiative transfer equation was justified for the spherical model of the atmosphere. The method is based on deformation of the coordinate system at every point in accordance with the spatial distribution of the refractive index in the medium. It was shown that in thus deformed coordinate system the differential RTE operator with refraction taken into account keeps the form characteristic of the system of ordinary spherical coordinates, while being simpler. It does not include explicitly any refraction terms and corresponds to the rectilinear trajectory of light rays.

It follows from the results obtained that for problems of the radiative transfer theory formulated for a spherical atmosphere with refraction taken into account there is no need in formal description of the real refraction curvature of light rays, as it can be taken into account in the RTE through the corresponding deformation of the coordinate system.

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