Nonlinear dynamics of directed acoustic waves in stratified liquids and gases

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The method has been developed for separation of the total hydrodynamic field into components: stationary one and those moving in opposite directions. Thus the set of equations breaks into three nonlinear equations for interacting components. New equations describing slightly linear interaction have been derived for the general form of the refined equation of state. The calculated nonlinear evolution of directed components shows that a directed wave in liquid keeps its direction even at high amplitudes of initial distortions. Solution of the problem is also interesting in view of a wide variety of problems with similar dispersion relation. The integral dispersion operator, whose form is determined by medium inhomogeneity, arises also in the theory of waveguide propagation of electromagnetic waves. The directed and stationary waves change the background temperature (and density), as well as the main refractive index and thus create moving or stationary areas of induced refraction and scattering of optical waves.

1. Problem of separation of directed waves and stationary component in the total hydrodynamic field

The most well-known methods of separation of wave distortions is separation by branches of the dispersion relation. This method allows separation of acoustic and gravitational waves as well as Rossby waves in gas and liquid.[1–3] A problem sometimes arises on separation of the total wave field in terms of the direction of propagation.[4] This problem, in turn, has many aspects, for example, separation of the total field into components at any instant of time, estimation of the wave energy for each component, construction of initial distortions for a particular type of distortion (dominant distortion for the nonlinear problem, which includes distortions of other types as well), and estimation of generation of the mean field and mutual influence of waves of different type.

The main idea within the framework of the linear theory is to collect the Fourier transform components corresponding to the same sign of ω and κ for the downward wave and those with the opposite sign for the upward wave (or left and right components for the homogeneous medium). First, we obtain the coupling equations for independent variables (for example, p, ρ, v) in the k-presentation and then collect them by the Fourier integral. For the stationary component, we collect components with $\omega = 0$. Thus, the total field of the linear problems breaks into the directed and stationary components, and projection operators are derived. Then the projectors are used to study the nonlinear dynamics.

2. Stratified liquid in the field of gravity

2.1. Dispersion relation and coupling equations

The set of equations of hydrodynamics is well-known: it includes the Newton's second law and the laws of conservation of energy and mass.[1,2] Let us use the problem geometry accepted in geophysical hydrodynamics. Let liquid be subject to gravitation giving rise to its density stratification:

$$\text{ä}v/\text{ä}t + v\,\text{ä}v/\text{ä}r = -(1/\rho)\,\text{ä}p/\text{ä}r - g,$$

$$\text{ä}\varepsilon/\text{ä}t + v\,\text{ä}\varepsilon/\text{ä}r = (-p/\rho)\,\text{ä}v/\text{ä}r, \qquad (1)$$

$$\text{ä}\rho/\text{ä}t + \text{ä}(\rho v)/\text{ä}r = 0.$$

Besides, we should add the equation of state $\varepsilon = \varepsilon(p, \rho)$. In Eq. (1) r is coordinate; t is time; ρ, p, ε, and v are density, pressure, inner energy per unit mass, and speed, respectively; g is the free fall acceleration. In the general case, the task is to express the inner energy in terms of the variables (p, ρ). For any liquid or gas, including stratified ones, distortion of the inner energy can be expanded into the Taylor series about small perturbations of the density and pressure. For slightly linear evolution it is sufficient to consider the series up to the second order of smallness:

$$\rho_0(\varepsilon - \varepsilon_0) = A(p - p_0) + B(\rho - \rho_0) + A_1(p - p_0)^2/p_0 +$$
$$+ B_1(\rho - \rho_0)^2/\rho_0 + D(p - p_0)(\rho - \rho_0)/\rho_0, \qquad (2)$$

ρ_0, p_0, ε_0 are unperturbed values.

Assume that the background density is exponentially stratified:

$$\rho_0(r) = \rho_{00}\exp(-r/h);$$

$$p_0(r) = p_{00}\exp(-r/h) = \rho_{00}\,gh\exp(-r/h),$$

where p_{00} and ρ_{00} are the surface values; h is the height of the "homogeneous" liquid (gas), i.e., the characteristic scale of vertical inhomogeneity; $p_{00} = \rho_{00}\,gh$ is conclusion of stationary solution of the linear analog of Eq. (1): $\mathrm{d}p_0(r)/\mathrm{d}r = -\rho_0(r)g$. For liquids it is taken that p_{00} is the inner pressure, rather than the static one (that is, atmospheric pressure over liquid). The corresponding components of the Fourier series for distortions in the exponentially stratified medium have the form

$$v'(k,t) = v'_0(k)\exp(r/2h)\exp[i(\omega t - kr)]\exp(\alpha r);$$

$$p'(k,t) = p'_0(k)\exp(-r/2h)\exp[i(\omega t - kr)]\exp(\alpha r);$$

$$p'(k,t) = \rho'_0(k)\exp(-r/2h)\exp[i(\omega t - kr)]\exp(\alpha r).$$

From the linear analog of Eq. (1) we obtain the dispersion relation

$$\omega^2 = k^2[(gh-B)/A] + g^2(A+1)^2/[4A(gh-B)],$$

where $\alpha = -(Agh + B)/[2h(gh-B)]$ or $\omega = 0$. Two signs of the frequency in solution of the dispersion equation and zero frequency mean three independent types of wave motion. It is proposed to separate the total wave motion into three independent parts: upward, downward, and stationary components, by introducing the complete orthogonal system of projection operators. This is convenient from the physical point of view, since the projection procedure can be applied to the field at any instant of time.

The main stages of derivation of the equations for the directed waves are the following.[4–7] First, from the linearized system (1) we have the following coupling equations for the Fourier components of distortions:

$$\rho'_0 = \rho_{00} v'_0 (ik + 1/2h - \alpha)/(i\omega);$$

$$p'_0 = \rho_{00} v'_0 [B(-ik - 1/2h + \alpha)] - gh(-ik + 1/2h + \alpha)/(Ai\omega).$$

Hereinafter, the distortions of hydrodynamic variables and the corresponding Fourier transforms are marked by primes. Then we obtain the coupling equations in terms of the variables (r, t) and collect the corresponding components by the Fourier integral: components with different signs of ω and k are collected for the upward moving wave, those with the same sign are collected for the downward wave, and components with $\omega = 0$ are collected for the stationary contribution. This procedure is described in detail in Refs. 5 and 6. The coupling equations in term of the variables (r, t) have the form

$$p_+ = L1 v_+;\quad \rho_+ = L2 v_+,\quad p_- = -L1 v_-;\quad \rho_- = -L2 v_-,$$

$$\rho_{\text{stat}} = L3 p_{\text{stat}}, \tag{3}$$

where $\rho_+ = \rho'_{\text{up}}\exp(-r/2h + \alpha r)$, $p_+ = p'_{\text{up}} \times \exp(-r/2h + \alpha r)$, $v_+ = v'_{\text{up}}\exp(r/2h + \alpha r)$, and so on. The integrodifferential operators $L1$, $L2$, and $L3$ have the form

$$L1 = \rho_{00}/\left(\pi\sqrt{(gh-B)/A}\right)\int_{-\infty}^{\infty} dr' \times \left\{-\frac{gh-B}{A}F_{AB}(r'-r)\,\ddot a/\ddot a r' + \frac{g(A-1)}{2A}F_{AB}(r'-r)\right\};$$

$$L2 = \rho_{00}/\left(\pi\sqrt{(gh-B)/A}\right)\int_{-\infty}^{\infty} dr' \times F_{AB}(r'-r)\left\{-\ddot a/\ddot a r' + \frac{g(A+1)}{2(gh-B)}\right\};$$

$$L3 = (1 - 2\alpha h - 2h\,\ddot a/\ddot a r)/(2gh),$$

where

$$F_{AB}(r) \equiv \frac{2}{\pi}\left\{I_0[rg(A+1)]/[2(gh-B)] - L_0[rg(A+1)]/[2(gh-B)]\right\};$$

I_0 and L_0 are the modified zero-order Bessel function and the Struve function, respectively. Then, using Eq. (3), from the first equation of the set (1) we also obtain the linear evolution equations for the upward and downward waves:

$$\ddot a v_\pm/\ddot a t \pm \sqrt{\frac{gh-B}{\pi^2 A}} \times \int_{-\infty}^{\infty}\left\{v_{\pm r'r'} - \frac{g^2(A+1)^2}{4(gh-B)^2}v_\pm\right\}F_{AB}(r'-r)\,dr' = 0. \tag{4}$$

The evolution equations can also be obtained for the variables $p_\pm$ and $\rho_\pm$ from the second and third equations of this set.

2.2. Projection operators

The integrodifferential matrix operators for the dynamics of ideal liquid and gas within the framework of the linear theory were obtained in Refs. 4 and 5. Now we have new equations with regard for expansion (2) of the equation of state:

$$P_\pm = \begin{pmatrix} 1/2 & \pm l1 & \pm l2 \\ \pm L1/2 & L1l1 & L1l2 \\ \pm L2/2 & L2l1 & L2l2 \end{pmatrix};$$

$$P_{\text{stat}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l3 & l4 \\ 0 & L3l3 & L3l4 \end{pmatrix}; \tag{5}$$

$$l1 = 1/\left(2\rho_{00}\pi\sqrt{(gh-B)/A}\right) \times \int_{-\infty}^{\infty} dr' F_{AB}(r'-r)\left\{-\ddot a/\ddot a r' + \frac{g(A+1)}{2(gh-B)}\right\},$$

3.1. Dispersion relation and coupling equations

In the case of homogeneous medium (liquid and gas), the equations take much more simpler form. Now the background density and pressure are coordinate-independent constants. The corresponding components of the Fourier series accept the form: $p'(k,t) = p'_0\exp[i(\omega t - kr)]$; $\rho'(r,t) = \rho'_0\exp[i(\omega t - kr)]$; $v'(r,t) = v'_0\exp[i(\omega t - kr)]$, where p'_0, ρ'_0, and v'_0 are constants, as well as the background values $p_0 = p_{00}$, $\rho_0 = \rho_{00}$. In the same manner, $\rho'_\pm = \pm L2v'_\pm$; $p'_\pm = \pm L1v'_\pm$. The dispersion relation has the form $\omega^2 = k^2[(p_0/\rho_0) - \hat A]/\hat A$ or $\omega = 0$. Since the medium dispersionless, then the group and phase velocities

$$c = [(p_0/\rho_0 - B)/A]^{1/2}. \tag{7}$$

The equations for projection operators can be derived directly from the above equations by the transition $h \to \infty$, $g \to 0$, $gh \to p_{00}/\rho_{00}$, $\alpha \to 0$. The new operators are simply factors: $L_1 = \rho_0 c$; $L_2 = \rho_0/\tilde n$; $l1 = 0$; $l2 = 1/(2L2)$; $l3 = 1$; $l4 = -L1/L2$. Now the new coupling between the stationary variables takes place: $p_{\text{stat}} = L\rho_{\text{stat}} = 0$, so $L = 0$. The operator chosen in such a way is inverse to $L3$, and formally we can set $L3$ equal to infinity. In a homogeneous medium it is natural to call the directed waves the right and left ones.

The linear evolution equations for the right and left waves have a very simple form: $\mathrm{d}v_\pm/\mathrm{d}t \pm c\,\mathrm{d}v_\pm/\mathrm{d}r = 0$; $v_\pm$, $p_\pm$, and $\rho_\pm$ are the corresponding parts of the standard variables v', p', and ρ'.

3.2. Projection operators

The projection operators possess the usual properties of orthogonal operators and have the form

$$P_\pm = \begin{pmatrix} 1/2 & \pm l/(2L1) & 0 \\ \pm L1/2 & 1/2 & 0 \\ \pm L2/2 & L2/(2L1) & 0 \end{pmatrix},$$

$$P_{\mathrm{stat}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -L2/L1 & 1 \end{pmatrix}.$$

3.3. Nonlinear evolution of the right and left waves

Following the scheme described in Section 2.3, we obtain the evolution equation for the right and left waves:

$$\mathrm{d}v_\pm/\mathrm{d}t \pm c\,\mathrm{d}v_\pm/\mathrm{d}r = (1/2 \quad \pm 1(2L1) \quad 0) \times$$

$$\times \begin{pmatrix} -v' \dfrac{ä}{är} v' + \dfrac{\rho'}{\rho_0^2} p' \\[2ex] -v' \dfrac{ä}{är} p' + \left(\dfrac{1}{A}\right) \dfrac{äv'}{är} \left[p'(-1 + D + 2A_1 c^2 \rho_0/p_0) + \right. \\ \left. + \rho'(p_0/\rho_0 + Dc^2 + 2B_1 + B)\right] \\[2ex] -v' \dfrac{ä}{är} \rho' + \rho' \dfrac{ä}{är} v' \end{pmatrix}$$

or for the case of self-action

$$äv_\pm/ät \pm c\, äv_\pm/är +$$
$$+ \frac{1 + (1 - D - 2A_1 c^2 \rho_0/p_0)/A - (p_0/\rho_0 + Dc^2 + 2B_1)/c^2}{2} \times$$
$$\times v_\pm\, äv_\pm/är = 0\,. \tag{8}$$

4. Applications of the theory

4.1. Dynamics of liquid or gas

Since the equation for the energy density is taken in the general form (linear coefficients of expansion $À$ and B enter into the operators, while the nonlinear coefficients $À_1$, $Â_1$, and D enter into the nonlinear column), the theory is applicable to a wide variety of liquids and gases. Besides these coefficients, we need to know only the background values of the density and pressure. For ideal gas, the equation is well-known which describes the inner energy $\varepsilon = p/[\rho(\gamma - 1)]$, $\gamma = C_p/C_v$, so all the coefficients of the expansion (2) are known as well: $À = 1/(\gamma - 1)$; $B = -p_0/\rho_0(\gamma - 1)$; $A_1 = 0$; $B_1 = -B$; $D = -A$. As an alternative for the case of a homogeneous medium, the equation $p/\rho^\gamma = p_0/\rho_0^\gamma$ is used or

$$p' = ñ^2 \rho' + \frac{c^2(\gamma - 1)}{2\rho_0} \rho'^2 \tag{9}$$

(Refs. 8 and 9). This gives the same square nonlinear terms.

Equation (9) follows from

$$\frac{\mathrm{d}p/\mathrm{d}t}{\mathrm{d}\rho/\mathrm{d}t} = \frac{\gamma p}{\rho}, \tag{10}$$

which, in turn, follows from the law of conservation of energy [the second equation of the set (1)], expression for the inner energy, and equation of continuity [the third equation of the set (1)]. Equation (9) follows from Eq. (10) only in the homogeneous medium.

For the stratified medium we have a completely different situation. In the exponentially stratified medium we should take into account the coordinate dependence of the background density and pressure and use, for example, $\dfrac{äp'}{ät} = c^2 \dfrac{ä\rho'}{ät} + \dfrac{v' p_0(\gamma - 1)}{h}$ in spite of the linear analog (9) $p' = ñ^2 \rho'$. Here $p_0(r) = p_{00} \exp(-r/h)$, as was defined in Section 2.1. For other type of stratification the coupling is different. In other words, the coupling of density and pressure distortions for the stratified medium ceases to be local. The expansion (2) is local, therefore the expansion coefficients can be used in a medium of any stratification.

4.2. Homogeneous liquid

After detailed consideration of the choice of the equation of state or its replacement in ideal gas, we pass to the more complicated case of a liquid for which only linear terms of the expansion (2) are available from reference sources. According to Eq. (2)

$$A = \frac{C_v k}{\beta V}, \quad B = -\left(\frac{C_p}{\beta V} - p_0\right)/\rho_0, \tag{11}$$

where V is the molar volume of the liquid; $\beta = -(ä\rho/äT)_{p = \mathrm{const}}/\rho$; $k = (ä\rho/äp)_{T = \mathrm{const}}/\rho$; $À$ and $Â$ are functions of the equilibrium variables (ρ_0, p_0). It is the practice to use Eq. (9) by analogy with ideal gas, and the parameter γ is assumed to be an empirical constant.[8,9]

In fact, $\gamma = (C_p/C_v)/(kp_0)$ (this equation follows from Eqs. (7) and (11), as well as Eq. (10), according to which $ñ = \gamma p_0/\rho_0$) is the function of density and pressure. For water at the temperature of 10°C and atmospheric pressure, $\gamma = 7.22$, whereas the approximation of temperature-independent constant γ is often used. For example, in Ref. 11 the value of 7.15 is taken. The method of expansion of the inner energy is much more accurate, since it allows one to find the local coupling between the distorted density and pressure as a function of background values. The coefficients of compressibility, thermal expansion, etc. can be taken from Ref. 12.

However, the nonlinear coefficients of Eq. (2) are not experimentally determined for liquids and are absent in the reference literature even for water. Let us use Eq. (9) when studying the nonlinear dynamics of liquids. In spite of the second equation of the set (1) and the equation of state we have

$$\frac{äp'}{ät} + \gamma p_0 \frac{äv'}{är} = -v' \frac{äp'}{är} - \gamma p' \frac{äv'}{är}. \tag{12}$$

The evolution equation for a directed wave with allowance made only for self-action can be found in the nonlinear acoustics of gases:

$$\ddot{a}v_{\pm}/\ddot{a}t \pm c\,\ddot{a}v_{\pm}/\ddot{a}r + \frac{\gamma+1}{2}\,v_{\pm}\,\ddot{a}v_{\pm}/\ddot{a}r = 0. \qquad (13)$$

The same equation can be derived by the method of slowly varying amplitude.[8,9] The analytical solution of Eq. (13) is also well-known. The dynamics of upward and downward waves for some types of distortions was discussed in Refs. 4, 5, and 10. The upward and downward waves keep their properties even for high initial amplitudes, for example, for the amplitude of speed of the initial conditions as high as 500 m/s. The initial distortions are constructed by the equations of the linear theory, whereas the dynamics was calculated by the well-known Lagrange nonlinear scheme.[4,5]

4.3. Exponentially density stratified liquid

In the general case, if all the coefficients of the expansion (2) are available, the nonlinear column can be presented as in Eq. (6). In the stratified medium it is not allowed to simply use the empirical equation (9) of coupling between density and pressure for the case of adiabatic process. Using ideal gas as an example, it was shown that the coupling ceases to be local and depends on the background stratification. In the case of known nonlinear coefficients of the expansion (2), the problem of slightly linear dynamics has an exact solution. Now we can only assume that Eq. (10) is valid for the liquid. Let us write down the evolution equation for the speed:

$$\ddot{a}v_{\pm}/\ddot{a}t \pm \sqrt{\frac{\gamma g h}{\pi^2}} \int_{-\infty}^{\infty} \left\{ v_{\pm r'r'} - \frac{1}{4h^2} v_{\pm} \right\} F(r - r')\, dr' =$$

$$= [1/2 \;\; 1/(2L1) \;\; 0]\, \exp(r/2h) \times$$

$$\times \begin{pmatrix} -v_s'\left(\frac{\ddot{a}}{\ddot{a}r} + \frac{1}{2h}\right) v_s' + \frac{\rho_s'}{\rho_{00}^2}\left(\frac{\ddot{a}}{\ddot{a}r} - \frac{1}{2h}\right) p_s' \\ -v_s'\left(\frac{\ddot{a}}{\ddot{a}r} - \frac{1}{2h}\right) p_s' - \gamma p_s'\left(\frac{\ddot{a}}{\ddot{a}r} + \frac{1}{2h}\right) v_s' \\ -v_s'\left(\frac{\ddot{a}}{\ddot{a}r}\,\rho_s' - \rho_s'\,\frac{\ddot{a}}{\ddot{a}r}\right) v_s' \end{pmatrix}. \qquad (14)$$

(10) markedly underestimates the amplitudes of speed, pressure, and density. This discrepancy is also caused by the use of Eq. (10) in spite of the equation of state (2) and the equations of conservation of energy and mass [the second and third equations of the set (1)].

4.4. Illustrations

Using Eqs. (13) and (14) we have calculated the linear and slightly linear dynamics of some types of initial distortions in stratified and homogeneous water. The following values were taken: $p_0 = 3050.9 \cdot 10^5$ Pa; $\rho_0 = 998.206$ kg/m^3, and $\gamma = 7.15$ (Refs. 11 and 12).

Figure 1 shows the slightly linear dynamics of the “saw-tooth” speed distortion in the left wave for the cases of homogeneous and stratified water. The role of dispersion is clearly seen: the distortion broadens and blurs. The wave amplitude in stratified water decreases. In spite of dispersion blurring, this is also caused by the fact that the plot is presented for the speed, while the equations for the directed wave were obtained for the variable $v \cdot \exp(-r/2h + \alpha r)$ [as was noted above, $\alpha = 0$ in the approximation (10)]. Figure 2 shows the linear and slightly linear dynamics of the downward speed wave in stratified water with allowance made only for wave “self-action.” The maximum speed achieves 200 m/s. Fifteen second after the beginning of the evolution, one can see characteristic nonlinear distortion of the wave and twisting of its trailing edge.

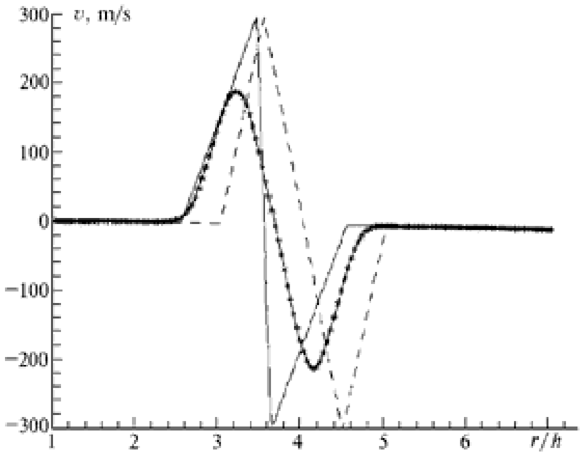

Fig. 1. Nonlinear evolution of the saw-tooth initial speed distortion for the wave propagating through homogeneous and stratified water: initial distortion (- - -); distortion in homogeneous water, 10 s after the beginning of evolution (—); distortion in stratified medium, 10 s after the beginning of evolution (***).

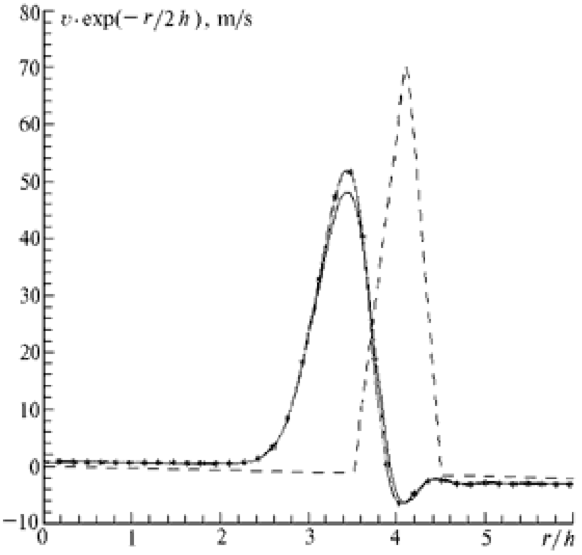

Fig. 2. Linear and nonlinear evolution of the initial speed perturbation for the downward wave propagating through stratified water: initial distortion (- - -); nonlinear distortion, 15 s after the beginning of evolution (—); linear distortion, 15 s after the beginning of evolution (***).

Conclusion

The projectors have been derived in the most general form dependent only on the equation of state in both homogeneous and exponentially stratified media. Any non-viscous liquid or gas can be considered as a medium. The projection operators serve to study the nonlinear dynamics and to separate the wave field into components: those moving in opposing directions and the stationary one. The form of the projectors depends on the dispersion relation, which also arises in the problems of waveguide propagation of electromagnetic waves.[13]

Application of the projectors allows studying problems of self-action of individual modes and their mutual influence, for example, generation of the mean field by the right or left wave. The projectors are also applicable in the problems with the boundary regime, as in the problems with initial conditions.

With the projection operators we have obtained the equation of slightly linear evolution of the directed mode with allowance made for only self-action in the most general form depending on the coefficients of serial expansion of the inner energy.

The directed waves and the stationary component change the parameters of the medium: density and pressure and, correspondingly, temperature and speed. This also changes the optical properties of the medium, in particular, the main refractive index. Another possibility is to determine a source, its power, and position from the induced change of the optical properties of the medium.

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