Artificial reference sources and anisoplanatism of fluctuations

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We analyze how the inhomogeneities along signal and reference atmospheric paths affect the efficiency of correction for phase distortions. Adaptive-optics systems operating with signals reflected from an object or scattered by atmospheric inhomogeneities as with reference ones are considered. The levels of residual distortions at various time lags in an adaptive system are studied. Calculations are performed for different scenarios of an optical experiment: ground paths and high-altitude (aircraft) horizontal paths. To describe altitude variations of the wind speed, the Bufton's model is used.

Introduction

Laser beams formed in the atmosphere are subject to the effect of turbulent fluctuations causing beam distortions.[1,2] To correct for turbulent distortions of optical beams, adaptive optical systems are used. Turbulent distortions themselves are independent of the object speed or the laser beam scanning speed, but requirements to an adaptive optical system (first of all, the frequency pass band[3]) in the case of a moving object are higher than those in the case of a stationary object. Adaptive optical systems usually use reference sources, whose radiation passes through the atmosphere in the direction opposite to the sounding radiation.

Since adaptive optical systems critically depend on the quality of the supplementary information used, the possibility of forming a reference source in the optical channel, in which the optical system itself operates, is very important. However, most often the optical scenario allows the reference source to be formed for a somewhat different optical channel or direction. This gives rise to manifestation of the so-called anisoplanatism of fluctuations.

Basic equations

Anisoplanatism in adaptive systems operating using a reference source is caused by different aberrations of the wave front along atmospheric paths for the corrected and reference beams. This difference is caused either by the difference in the path lengths of the optical wave because of purely geometric factors (different paths or different radiation divergence of the corrected and reference beams) or by the time lag.[4–8] In many cases, the causes giving rise to these factors are similar and, what's more, the methods of their mathematical representation are similar as well.

Let us consider the problem of compensation for turbulent aberrations in the approximation of a phase screen generated on the path at the distance x. It is assumed that turbulent fluctuations are delta-correlated along the direction of propagation. It follows from here that to calculate fluctuations, one should perform integration along the variable x. Assume that a random screen is generated in the plane x; $\varphi(\boldsymbol{\rho}, x, t)$ are phase fluctuations at a time t and a point $\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the cross coordinate. Using the Taylor's "frozen turbulence" hypothesis, we can relate the parameters of the phase screen at the time t and $t + \tau$ as follows:

$$\varphi(\boldsymbol{\rho}, x, t+\tau) = \varphi(\boldsymbol{\rho} - \mathbf{V}\tau, x, t) = \varphi(\boldsymbol{\rho} - \mathbf{V}t - \mathbf{V}\tau, x, 0), \quad (1)$$

where τ is the time lag; $\mathbf{V}$ is the wind velocity vector. The similarity of the mathematical description of the angular anisoplanatism and the anisoplanatism caused by the time lag follows from Eq. (1).

The optical scenario of the experiment is characterized by the coordinates of the target and the reference source – beacon. We use here the angular coordinates of the target θ_T and beacon θ_b, which are the ratio of the linear coordinates and the distance between the observation plane and the plane of the target (or beacon). The angular coordinates of the beacon and the direction of the system's optical axis are described by the variables $\theta_b(t)$ and $\theta_A(t)$. The angular coordinate $\theta_T(t)$ for an actual object (target) being at the distance L and moving at a speed $v = M \cdot 330$ m/s in the cross direction varies as

$$\theta_T(t) = vt / L = \omega t, \quad (2)$$

where ω is the angular frequency of the object motion; M is the Mach number. If the beacon is formed directly by reflection from the object itself, then the angular variable $\theta_b(t) = \theta_T(t)$. If some arbitrary displacement of the object takes place, then, because of the finite speed of light c and the presence of some lag in the adaptive system τ_d, to provide correcting for fluctuations, the Rayleigh beacon should be formed in advance, that is, it should be formed at that point, where the object will be at the time $t + \tau_d$. Then

$$\theta_b(t) = \theta_T(t) + 2v\tau_c / L + v\tau_d / L =$$

$$= \theta_T(t) + 2v/c + v\tau_d/L, \quad \tau_c = L/c. \tag{3}$$

In the general case

$$\theta_b(t) = \theta_T(t) + \eta\,(2v/c + v\tau_d/L), \tag{4}$$

where η is the advance parameter of the beacon; $\eta > 0$, when the beacon advances the object, or $\eta = 0$, if the beacon does not advance the object.

When choosing the direction of the axis of the adaptive optical system, the lag associated with the finite speed of light should also be taken into account. Thus,

$$\theta_A(t) = \theta_T(t) + v\tau_c/L = \theta_T(t) + v/c. \tag{5}$$

As a result, the reference wave emitted at the time t acquires, when crossing the phase screen, aberrations described by the function

$$\varphi_b(\boldsymbol{\rho}) = \varphi[\boldsymbol{\rho} + \theta_b(t)x,\ t + (L - x)/c] = \varphi[\boldsymbol{\rho} + \theta_b(t)x - (L - x)V/c,\ t]. \tag{6}$$

Here we assume that the time of propagation of the reference wave to the phase screen is $(L - x)/c$.

Rayleigh beacon

At the time $t + L/c$ the reference wave achieves the aperture of the adaptive optical system, and at the time $t + L/c + \tau_d$ the correcting surface is formed with the use of the information carried by this reference wave.

The controlled laser beam comes to the phase screen at the time $t + L/c + \tau_d + x/c$ and acquires the following aberrations:

$$\varphi_A(\boldsymbol{\rho}) = \varphi[\boldsymbol{\rho} + \theta_A(t + L/c + \tau_d)x,\ t + L/c + \tau_d + x/c] = \varphi[\boldsymbol{\rho} + \theta_A(t + L/c + \tau_d)x - V(L/c + \tau_d + x/c),\ t]. \tag{7}$$

Comparing Eqs. (6) and (7) one can see that the residual error of correction is caused by the relative displacement of the phase screen. This displacement is equal to

$$\Delta = [\theta_A(t + L/c + \tau_d) - \theta_b(t)]\,x - V\,(\tau_d + 2x/c). \tag{8}$$

Substituting it into Eq. (5), we obtain

$$\Delta = [v/L\,(t + \tau_d) + 2v/c - \theta_b(t)]\,x - V\,(\tau_d + 2x/c) \tag{9}$$

and, using Eq. (4), we find

$$\Delta = v(\tau_d x/L + 2x/c)\,(1 - \eta) - V\,(\tau_d + 2x/c). \tag{10}$$

When $\tau_d = 0$, Eq. (10) becomes simpler, namely

$$\Delta = 2x/c\,(v\,(1 - \eta) - V). \tag{11}$$

It should be noted that this relative displacement Δ can be reduced to zero through the proper choice of the parameter η:

$$\Delta = 0 \text{ for } \eta = \eta_0 = 1 - V/v. \tag{12}$$

If the wind speed is equal to the object speed, i.e., $V = v$, then the optimal value of the parameter η equals zero, i.e., the beacon formed by the signal reflected from the object is optimal.

If $v \gg V$, then the optimal value of η approaches unity, i.e., just the anticipatory Rayleigh beacon is optimal. In the case that the object moves in the direction opposite to the wind, the optimal value of η_0 is larger than unity.

At a considerable time lag τ_d, the displacement Δ can be reduced to zero, if we take

$$\eta_0 = 1 - \frac{V}{v}\,\frac{\tau_d + 2x/c}{\tau_d x/L + 2x/c}. \tag{13}$$

In the case that the turbulence is concentrated near the object, that is, $x = L$, we have

$$\eta_0 = 1 - V/v. \tag{14}$$

In this situation, the optimal value of the parameter η is independent of the time lag. In the opposite situation, when the value of the coordinate x approaches zero, we have

$$\eta_0 \approx 1 - \frac{V}{v}\,\frac{1}{x}\,\frac{\tau_d}{\tau_d/L + 2/c}, \quad x \ll \frac{\tau_d c}{2}, \tag{15}$$

or

$$\eta_0 \approx 1 - \frac{L/v}{x/V}, \quad \tau_d \gg 2L/c = 2\tau_c. \tag{16}$$

In the area adjacent to the aperture (coordinate x is small), we can omit unity in the latter equation, thus obtaining that

$$\eta_0 \approx -\frac{L/v}{x/V}, \tag{17}$$

that is, if the wind and the object have the same direction, then the advance parameter tends to negative infinity, and if the object moves in the direction opposite to the wind, then it tends to the positive infinity.

In the general case, turbulence is distributed all over the path, and we have to consider a set of phase screens. Moreover, the wind speed may depend on the coordinate x, and this means that phase screens should move with different speed.[9]

Spatial filtering function

In the general case, optimization by anticipatory re-aiming of the beacon is rather difficult. Moreover, in the majority of realistic situations, the intensity of the Rayleigh beacon is too low to provide for the phase correction. However, in some cases the anticipatory re-aiming of the beacon gives some advantages.

To estimate the correction efficiency, we should calculate the variance of residual fluctuations. This variance is the path integral of some function depending on the intensity of turbulence $C_n^2(x)$ and on the displacement $\Delta(x)$.

The needed equation can be derived from the well known equation describing the angular anisoplanatism, since in both cases the source of residual errors is the same. For the angular anisoplanatism, the variance of residual phase errors was presented in Ref. 8 as

$$\sigma_\varphi^2 = (\theta / \theta_0)^{5/3} = \theta^{5/3}\, 2.91k^2 \int_0^L C_n^2(x) x^{5/3} \mathrm{d}x\ . \quad (18)$$

Since the product of θ by x is the relative displacement of the trajectories of the reference and basic beams, we can obtain the following equations:

$$\sigma_{\Delta\varphi}^2 = 2.91k^2 \int_0^L C_n^2(x)(\theta x)^{5/3} \mathrm{d}x =$$

$$= 2.91k^2 \int_0^L C_n^2(x) \Delta^{5/3}(x) \mathrm{d}x. \quad (19)$$

Note that for the phase screen, i.e., a thin turbulent layer characterized by the coherence length

$$r_0^{-5/3} = 0.423k^2 C_n^2(x)\delta x\ , \quad (20)$$

the equation for the variance has the following form:

$$\sigma_{\Delta\varphi}^2 = 6.88(\Delta / r_0)^{5/3}\ . \quad (21)$$

This equation coincides with the phase structure function written for the difference coordinate Δ. The variance of the residual error calculated by this equation accounts for the phase fluctuation component determined as a constant lag (piston). Since this component does not affect the correction efficiency, the efficiency is overestimated.

Equation (19) becomes more accurate, when the ratio Δ / D (here D is the aperture diameter) decreases. The equation for the variance of phase fluctuations without the constant component of field fluctuations is given in Refs. 4, 10, and 11. With the designations used in this paper, this equation can be written as

$$\sigma_{\Delta\varphi}^2 = 2.91k^2 \int_0^L C_n^2(x) D(x)^{5/3} f[\Delta(x) / D(x)] \mathrm{d}x\ , \quad (22)$$

where

$$f(\alpha) = 0.896 \int_0^\infty u^{-8/3} \mathrm{d}u \left[1 - J_0(2\alpha u)\right] \left(1 - 4\frac{J_1^2(u)}{u^2}\right) \quad (23)$$

is the spatial filtering function,[9] and the function $D(x)$ is a projection of the aperture diameter onto the plane x.

It should be taken into account that the reference beam diverges, while the direct (basic) beam is focused. For beams with large diameters and small beacons, the equation for a spherical wave can be used, namely,

$$D(x) = D\,(1 - x / L). \quad (24)$$

At small values of the argument, the filtering function

$$f(\alpha) = \alpha^{5/3} \quad (25)$$

and Eq. (22) transforms into Eq. (19). At large values of the argument, the filtering function saturates and the resulting variance of the error of phase correction achieves the level twice as high as the level achieved by the system without a correction, that is,

$$\sigma_{\Delta\varphi}^2(\Delta / D \to \infty) = 2 \cdot 1.03(D / r_0)^{5/3}\ . \quad (26)$$

The function $f(\alpha)$ is shown in Fig. 1. Dots correspond to the values obtained from numerical integration. The solid curve is the result of approximation of the calculated results by a nine-order polynomial. The approximation was performed in the logarithmic coordinates, that is,

$$\log f(\alpha) = \sum_{n=0}^{9} a_n (\log \alpha)^n\ . \quad (27)$$

Using the approximation (27), we calculated the variance of the correction error for different paths. The coefficients of the approximation are given below in the Table.

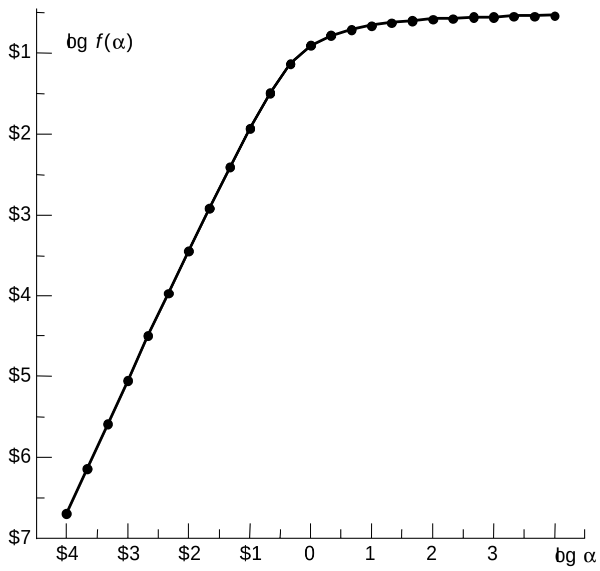

Fig. 1. Spatial filtering function $f(\alpha)$.

For the case of correction with the use of the reflected signal, let us consider the effect of a time lag[10] caused by the finite speed of light, i.e., when $\eta = 0$ and $\tau_d = 0$. As the object speed increases, the residual error of correction increases too. Therefore, there exists a limiting object speed corresponding to a given level of the residual error. Assume that the residual error equals 10% of its level in the absence of correction, that is,

$$\sigma_{\Delta\varphi}^2 < \frac{1}{10}\sigma_\varphi^2 = 0.1 \cdot 1.03(D / r_0)^{5/3}\ . \quad (28)$$

Table

a) if −4 < log α < 0, then

$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
− 0.90300263	+ 0.54630991	− 0.0075005029	+ 1.9852349	+ 3.0232371
$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
+ 2.2959981	+ 1.0072076	+ 0.25730477	+ 0.035491319	+ 0.0020411442

b) if 0 < log α < 4, then

$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
− 0.90300481	+ 0.46973638	0.41229668	+ 0.29897322	− 0.15200895
$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
+ 0.042862686	0.0024713447	−0.0020118738	+5.33356E−4	−4.1574399E−5

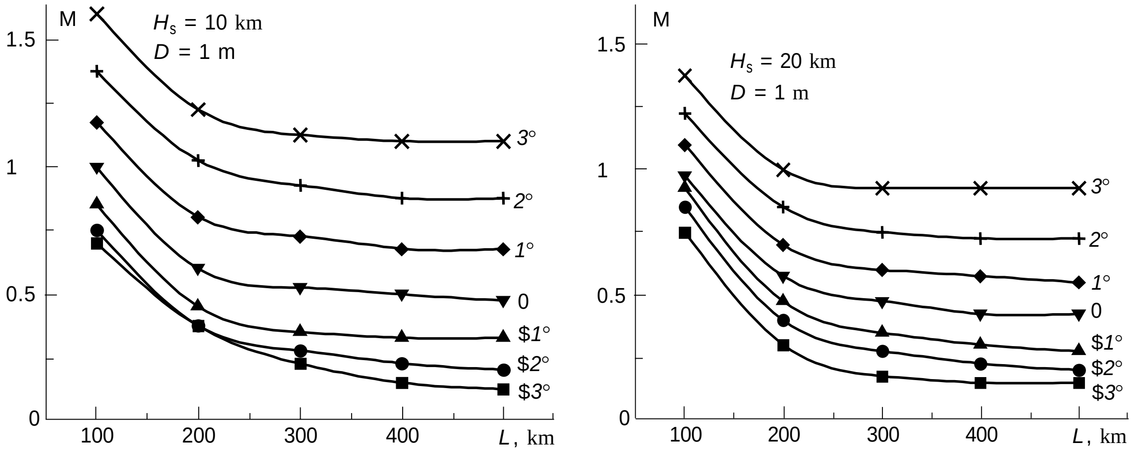

Fig. 2. Maximum object speed (measured in Mach numbers) at high-altitude atmospheric paths under conditions of adaptive correction using reflected signal as a reference one. The wind speed is 10 m/s, and the residual error is 10%. Time lag is absent in the system. Horizontal propagation of radiation (0), paths directed to the lower hemisphere (−*1*°, −*2*°, −*3*°), and paths directed to the upper hemisphere (*1*°, *2*°, *3*°).

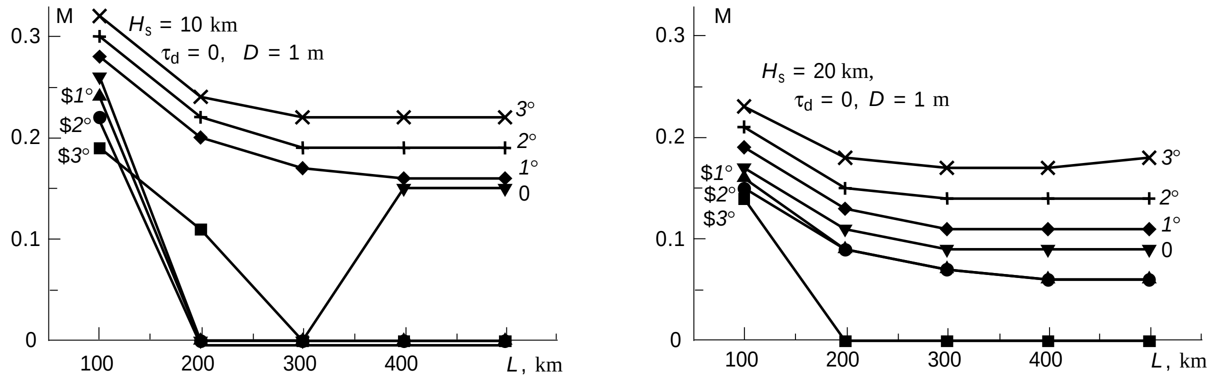

Fig. 3. Maximum object speed at high-altitude atmospheric paths under the conditions of adaptive correction using reflected signal. Calculation was made with the use of the Bufton's model for the wind velocity V_g = 5 m/s, residual error is 1%. Horizontal propagation of radiation (0), paths directed to the lower hemisphere (−*1*°, −*2*°, −*3*°), and paths directed to the upper hemisphere (*1*°, *2*°, *3*°).

The maximum object speed calculated for this level of the residual error is depicted in Figs. 2 and 3. The calculations were performed assuming constant wind speed of 10 m/s. The direction of the object motion coincided with the direction of the wind. Consequently, the maximum object speed usually does not exceed the speed of sound. For the paths with the negative tilt angle as measured from the horizon (this can take place only at elevated paths), this value is no larger than the half speed of sound. It should be noted that, according to the condition (28), high efficiency of correction can be achieved only at small ratio D/r_0. For example, for $0.103(D/r_0)^{5/3} < 1$ we obtain the condition $(D/r_0) < 3.9$.

Application of the Bufton's model

Similar results were obtained with the use of the Bufton's model[8] of the vertical distribution of the wind speed; the results are depicted in Figs. 4 and 5. The Bufton's wind model, as known, is described by the following equation:

$$V(h) = V_g + 30\exp\left[-\left(\frac{h-9400}{4800}\right)^2\right], \qquad (29)$$

where V_g is the model parameter corresponding to the wind speed at the ground level, h is the current height above the ground, in m. In calculations it was assumed that V_g = 5 m/s.

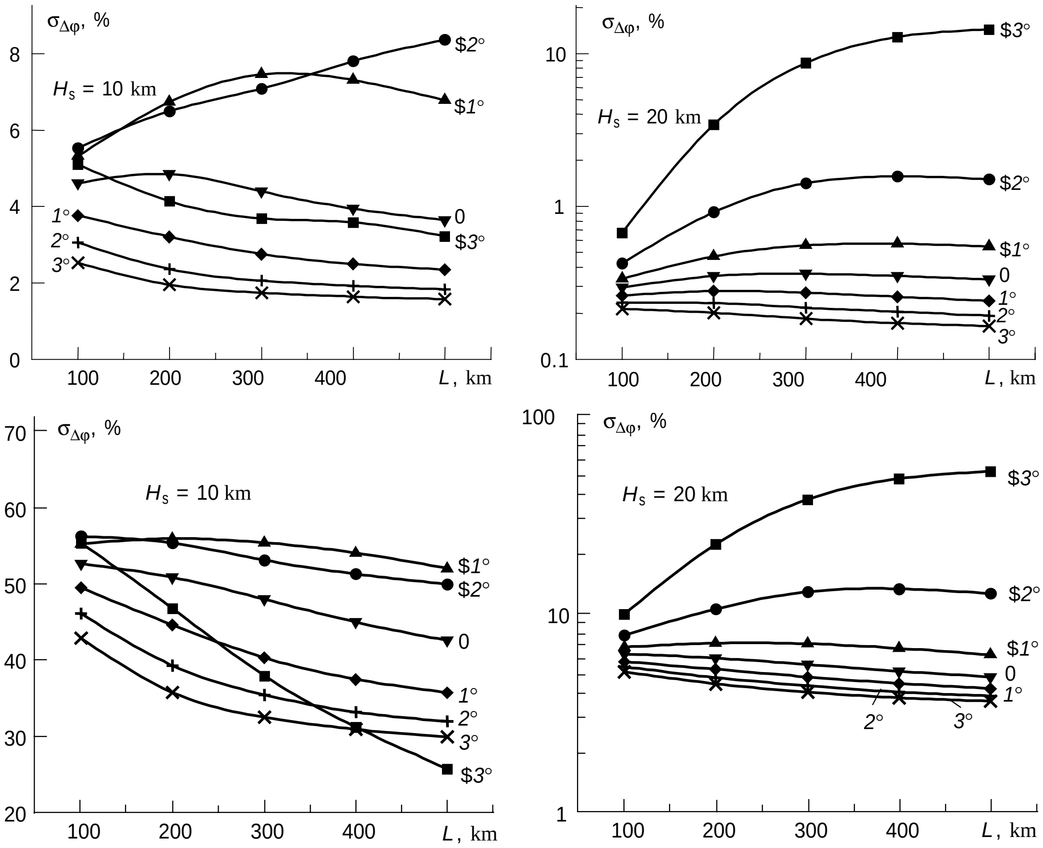

Fig. 4. Normalized residual error of phase correction with the use of Rayleigh beacon. The time lag in the system was 1 ms (upper row) and 10 ms (lower row). The Bufton's wind V_g = 5 m/s; D = 1 m. Horizontal propagation of radiation (0), paths directed to the lower hemisphere ($*1*°, $*2*°, $*3*°), and paths directed to the upper hemisphere (*1*°, *2*°, *3*°).

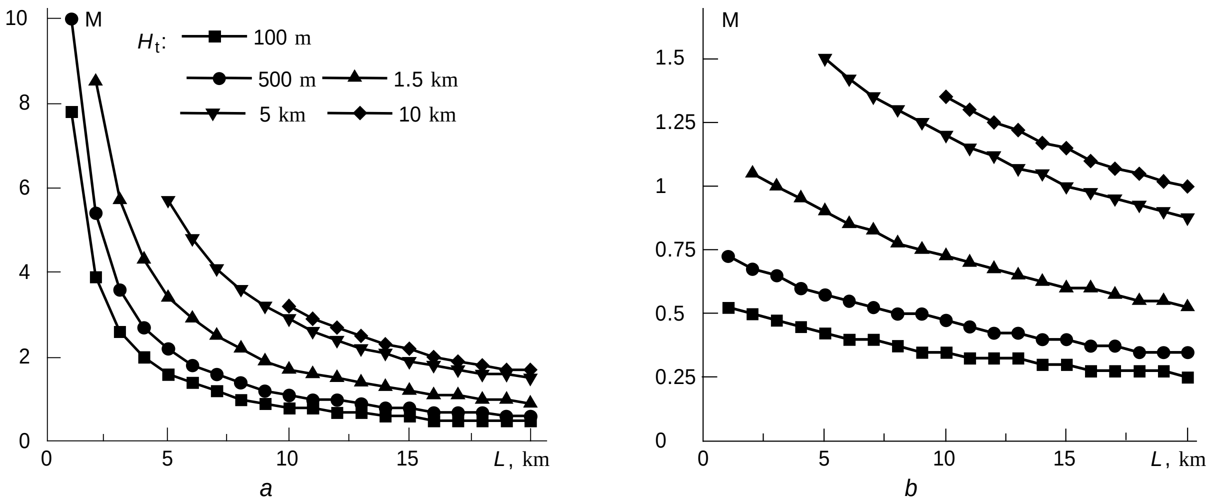

Fig. 5. Maximum allowable object speed (in Mach numbers) at short atmospheric paths for the adaptive system operating against the reflected signal. Residual error is 1%. The time lag in the system is τ_d = 0 (*a*) and τ_d = 1 ms (*b*); H_t is the target height; H_s = 5 m; the Bufton's wind V_g = 5 m/s; D = 0.5 m.

In this case, the stricter condition

$$\sigma^2_{\Delta\varphi} < \frac{1}{100}\sigma^2_{\varphi} = 0.0103(D\,/\,r_0)^{5/3} \qquad (30)$$

is super-overestimating for the residual phase error. As can be seen, for some paths the correction error exceeds this limit even at the zero object speed. Under the most favorable conditions, the maximum object speed should be below 50–100 m/s. When the object moves in the direction opposite to the wind, this restriction becomes even stricter.

It follows from the above data that the correction with the use of the signal reflected from the object itself has limited capabilities.

Let us consider the correction with the use of the Rayleigh beacon. In this case, the correction error can be minimized by selecting a proper value of the re-aiming parameter η. Since the case that the object speed is much higher than the wind speed is more important for us, the optimal value of the re-aiming parameter η is close to unity. It follows from Eq. (10) that the correction error is fully determined by the wind speed profile and the time lag τ_d and independent of the object speed.

As an example, let us consider the result obtained with the use of the Rayleigh beacon (see Fig. 4). The plots show the variances of the residual phase aberrations normalized to the variance of the same fluctuations in the absence of correction. The time lag of the adaptive system is equal to 1 and 10 ms.

The calculations show[9] that the zero time lag corresponds to the maximum efficiency of the adaptive system operating with application of the Rayleigh beacon, as well as for the system using the reflected signal, when the object speed equals zero. It can be seen that even for the longest paths the relative error of the phase correction is no larger than 1%. Thus, for the Bufton's wind model, the residual error proves to be almost an order of magnitude smaller, if the optical system is placed at the altitude of 20 km. In the Bufton's model, as known, the maximum wind speed (35 m/s) takes place at the altitude of 10 km.

As the time lag increases up to 1 ms, the residual phase error becomes equal to several percent. For the time lag of 10 ms, the error increases up to 30–60% for the system altitude H_s = 10 km and up to 5–10% for the altitude H_s = 20 km. Thus, we have that for rather high correction the time lag should be decreased down to 1 ms and for the efficient operation in the whole range of the considered paths the lag should be within 0.1 ms.

Similar calculations were made for the ground atmospheric paths as well. The results corresponding to correction with the error smaller than 1% are shown in Fig. 5, wherefrom we obtain the maximum allowable object speed at the zero time lag of the adaptive system. The maximum object speed is less than 2–3 M roughly in every second case. When the time lag is about 0.1 ms, the maximum object speed varies from 0.5 to 1.5 M. Thus, at these paths the use of a beacon based on reflection again leads to the strict conditions on the object speed and the speed of adaptive control.

Taking the advance parameter equal to unity, let us consider the control based on the Rayleigh beacon. For this beacon, residual distortions are independent of the object speed and fully determined by the wind speed profile and the time lag of the adaptive system. The calculated results normalized to the residual error at the time lag of 1 ms are shown in Fig. 6. In the whole interval, we obtain the residual error less than 1%, and this error is almost independent of the path length.

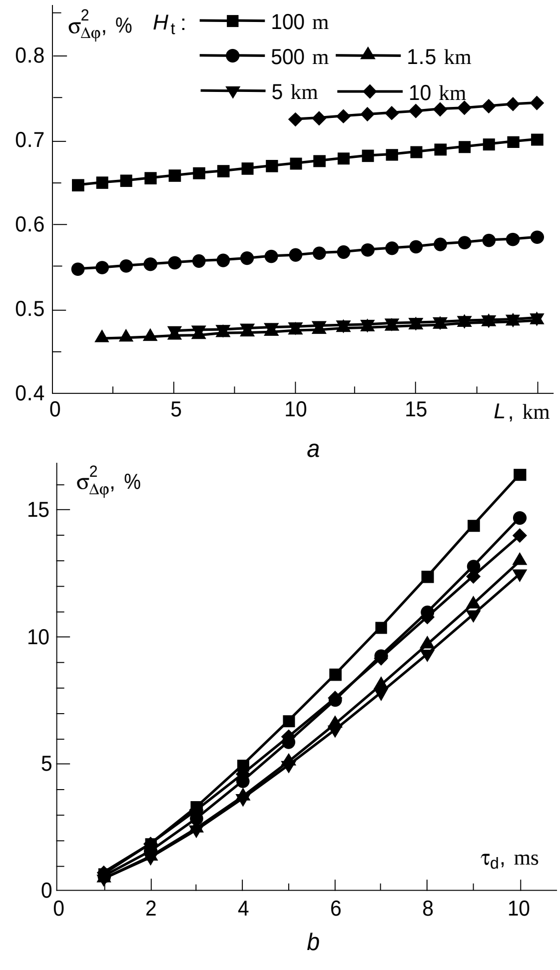

Fig. 6. Residual distortions as a function of the path length for the time lag of 1 ms (*a*) and as a function of the time lag in the feedback circuit (*b*) of the adaptive system for different initial altitudes of the source. In both cases the Rayleigh beacon was used as a references source. H_s = 5 m; the Bufton's wind V_g = 5 m/s; D = 0.5 m.

To study the dependence of the residual error on the time lag, let us consider a 10-km long path (Fig. 6*b*). It can be seen from Fig. 6*b* that the increase of the time lag up to 10 ms increases the residual error up to almost 15%. Therefore, the maximum allowable time lag in the adaptive system is 1–2 ms.

Conclusions

Application of a signal both reflected from an object and scattered by the atmospheric inhomogeneities (Rayleigh signal) as a reference beacon makes it possible creating a realistic adaptive optical system. This system can operate along both low and high-altitude (aircraft) paths. Since the corrected and reference signals do not coincide, a rather high level of correction, i.e., the small value of residual phase distortions caused by anisoplanatism, can be achieved only with a fast enough adaptive system, in which the total time lag does not exceed 1$2 ms.

Acknowledgments

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