Modeling of spatial distribution of the atmosphere-scattered radiation polarization coefficient on the base of complete analytical solution of the vector transfer equation

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An effective method for analytical calculation of polarization characteristics of radiation multi-scattered by turbid medium is considered. The efficiency of the method is in the assumption that the complete solution of vector radiative transfer equation (VRTE) is the superposition of the anisotropic and smooth parts, computed separately. The vector small-angle modification of the spherical harmonics method (SGM) is used to evaluate the anisotropic part, containing all singularities of the solution; and the matrix discrete ordinates method is used to obtain the smooth one, adding SGM to complete VRTE solution. The process of polarized radiation transfer is considered in a horizontally infinite layer of an arbitrary optical depth, irradiated by a plane mono-directed radiation source. Calculation examples are given for natural nonpolarized radiation. However, advantages of the suggested method carry over an arbitrary (including three-dimensional) geometry of scattering medium, arbitrary form of source, and radiation polarization state.

Introduction

It is known that the radiation polarization state, described by the four-component Stocks vector-parameter (SVP), natural from the photometric point of view, contains all the information about a sensing object, accessible to optical investigation methods.[1,2] Nevertheless, today the amount of researches in scalar approximation, i.e., with accounting for only radiation brightness (first SVP component) essentially exceed polarimetric ones. This is concerned with a comparatively small number of working polarimeters, which can be explained, in its turn, by two main reasons. First, this is the design problem in constructing the polarimeters necessary to detect quite weak SVP polarization components. However, the technology state-of-the-art allows the solution of the design problems. Hence, the second reason is principal: a lack of reliable mathematical model, accounting for polarization effects and serving as a reliable base for interpreting the polarization measurements (see, e.g., SPIE, V. 5888, "Polarization Science and Remote Sensing II," 2005, where a wide spectrum of experimental works is given against a small number of theoretical ones).

Like in the scalar case, the vector (polarization) radiation transfer model should be quite effective, i.e., possess a high rate of the convergence of the boundary problem of vector radiation transfer equation (BP VRTE) solving to the exact one. The model also should allow calculations for strongly anisotropic phase functions, characteristic of many natural formations (clouds, ocean, cosmic dust), for different optical depths τ (tenth fractions for clear atmosphere, and units and tens for clouds, shallow waters, and semi-infinite medium – the ocean), as well as for an arbitrary irradiation zenith angle θ_0. Accounting for scattering multiplicity is obligatory. An analytical form of the solution allows the calculation optimization and some simplification of solution of the inverse problems. An arbitrary state of the source polarization widens model applicability and makes it suitable for operation both with passive polarimeters and lidars.

In this work, we suggest a model of polarized radiation transfer in a turbid (scattering and absorbing) medium layer, irradiated by a plane mono-directed (PM) radiation source in the arbitrary direction $\hat{\mathbf{l}}_0$. The above-mentioned requirements were taken as model criteria.

Definition of anisotropic part

Let us take the following designation: «→» is the four-element column vector; «↔» is the sixteen-element Mueller matrix; Λ is the photon survival probability in one scattering event (single scattering albedo); θ and φ are the zenith and horizontal angles, respectively;

$$\mu = \cos\theta; \quad \hat{\mathbf{l}} = [\sqrt{1-\mu^2}\cos\varphi, \sqrt{1-\mu^2}\sin\varphi, \mu]$$

is the unit spatial direction. Designate SVP and its components as $\vec{\mathrm{L}} = [I\,Q\,U\,V]^{\mathrm{T}}$, where "T" is the transposition operation; I is the total beam brightness; Q and U are the energy components determining the linear polarization degree and

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reference plane position; V determines the degree of ellipticity.

In this work, we deal with radiation scattering, however, the suggested method is also effective for particle physics, where consideration of spin causes changeover to vector problems,[2] and the scattering anisotropy is usually essentially higher than the atmospheric one.

One of the main questions in solution of radiation transfer problems, both scalar and vector, is accounting for singularities of the BP for transfer equation with corresponding conditions. These singularities follow from the ray approximation in description of a medium-scattered radiation field. A mathematical singularity for a PM-source is caused by direct non-scattered radiation, expressed by the Dirac δ function. In analytic form, the singularity requires the infinitive number of terms of basic functions expansion, hence, it could not be calculated analytically. Chandrasekhar suggested to consider the scattered radiation field in a two-component form, i.e., the direct non-scattered δ-singularity, forming the source functions in the transfer equation, and the scattered light.[3] However, the scattered light for real strongly anisotropic turbid media is described by a "sharp" function of angular light distribution, which requires the significant number of expansion terms for its definition. This results in an increase of calculation time and, in a number of cases, in deterioration in solution conditionality.

In this work we apply the technique widely used in the scalar approximation[4–7] and present the target vector field as a superposition of the most anisotropic part, containing the direct radiation, and a smooth part (designated by the subscript R – "regular") which is regular in contrast to the singularity-containing anisotropic part. The scattering anisotropy is concentrated within "small" scattering angles relative to radiation incidence onto an elementary scattering volume (here the term "small angles" is historical and does not reflect real range of the method applicability). Taking the above-mentioned into account, write

$$\vec{\mathrm{L}}(\tau,\hat{\mathbf{l}}) = \vec{\mathrm{L}}_{\mathrm{R}}(\tau,\hat{\mathbf{l}}) + \vec{\mathrm{L}}_{\mathrm{SA}}(\tau,\hat{\mathbf{l}}). \quad (1)$$

Basis functions for polarized radiation transfer problems are generalized spherical functions (GSF); their properties, the addition theorem, and recurrence relations, for which

$$\ddot{\mathrm{Y}}_m^k(\mu) = \mathrm{diag}\left[P_{m,+2}^k(\mu);\ P_{m,+0}^k(\mu);\ P_{m,-0}^k(\mu);\ P_{m,-2}^k(\mu)\right]$$

are known.[8]

Write the standard GSP-presentation of the sought SVP spatial distribution and scattering matrix $\ddot{x}$ (for the Circular Polarization (CP) basis[9] used in the vector transfer theory for scattering integral expansion):

$$\vec{\mathrm{L}}_{\mathrm{CP}}(\tau,\hat{\mathbf{l}}) = \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\ddot{\mathrm{Y}}_m^k(\mu)\vec{\mathrm{f}}_m^k(\tau)\exp(im\varphi);$$
$$\left[\ddot{x}(\hat{\mathbf{l}}\hat{\mathbf{l}}')\right]_{r,s} = \sum_{k=0}^{\infty}(2k+1)x_{r,s}^k(\tau)\mathrm{P}_{r,s}^k(\hat{\mathbf{l}}\hat{\mathbf{l}}'). \quad (2)$$

The anisotropic part is calculated in the vector small-angle modification of the spherical harmonics method.[10] VSHM (similar to SHM,[4–7] which is scalar approximation of VSHM) is based on a continuous-function approximation of the spatial spectrum $\vec{\mathrm{f}}_k^m$, discrete relative to the zenith index k (order of the GSP of $P_{m,n}^k(\mu)$), of the sought vector brightness field. Smoothness of the spatial spectrum due to solution singularity and scattering anisotropy allows one to bound by two terms of the Taylor expansion of $\vec{\mathrm{f}}_k^m$ by the harmonic number k:

$$\vec{\mathrm{f}}^m(\tau,k\pm1) \approx \vec{\mathrm{f}}^m(\tau,k) \pm \frac{\partial\vec{\mathrm{f}}^m(\tau,k)}{\partial k}. \quad (3)$$

This assumption along with the scattering integral expansion on the base of GSF addition theorem in complex circular basis, recurrence relations for GSP, and equation (2) allow one to obtain a simple analytic expression to calculate the anisotropic part in the form of matrix exponent

$$\vec{\mathrm{L}}_{\mathrm{VSHM}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0) = \sum_{m=-2}^{0,2}\sum_{k=0}^{\infty}(2k+1)\ddot{\mathrm{P}}_k^m(\hat{\mathbf{l}}\hat{\mathbf{l}}_0)\times$$
$$\times\exp\left[-(\ddot{1}-\Lambda\ddot{\chi}_k)\tau/\mu_0\right]\vec{\mathrm{f}}_k^m(0)\exp(im\varphi)/4\pi, \quad (4)$$

where the coefficients of scattering matrix expansion in the real energy SP basis (Stokes Polarization) is $\ddot{\chi}_k = \ddot{\mathrm{T}}_{\mathrm{SC}}\ddot{x}_k\ddot{\mathrm{T}}_{\mathrm{CS}}$ and matrix polynomials have the form

$$\ddot{\mathrm{T}}_{\mathrm{SC}}\ddot{\mathrm{Y}}_k^m(\mu)\ddot{\mathrm{T}}_{\mathrm{SC}}^{-1} = \ddot{\mathrm{P}}_k^m(\mu) =$$
$$= \begin{bmatrix} P_{m,0}^k(\mu) & 0 & 0 & 0 \\ 0 & R_m^k(\mu) & -iT_m^k(\mu) & 0 \\ 0 & iT_m^k(\mu) & R_m^k(\mu) & 0 \\ 0 & 0 & 0 & P_{m,0}^k(\mu) \end{bmatrix}. \quad (5)$$

Here $\ddot{\mathrm{T}}_{\mathrm{SC}}$ is the known matrix of transition from complex CP to energy (Stokes) SP basis[9] and polynomials

$$R_m^k(\mu) = 0.5[P_{m,2}^k(\mu) + P_{m,-2}^k(\mu)];$$
$$T_m^k(\mu) = 0.5[P_{m,2}^k(\mu) - P_{m,-2}^k(\mu)]$$

are similar to Ref. 11.

Vectors $\vec{f}_k^m(0)$ in Eq. (4) are known from the boundary conditions.[10] VSHM (4) allows the calculation within the virtually whole forward hemisphere (the set of directions in the half-space, containing directions of radiation incidence on a layer). The reliability domain naturally depends on medium parameters. The higher is the brightness body sharpness (i.e., the stronger is the scattering anisotropy and the smaller is the layer optical depth), the wider is the domain, where VSHM gives a sufficiently good result. Note, that VSHM (4) includes not only the anisotropic part of scattered radiation, but also the singularity of BP VRTE. It is possible to expand the matrix exponent in Eq. (4) analytically.[12,13] However, modern math packages, e.g., The MathWorks Matlab®, allows the solution to be obtained directly from Eq. (4). Computation of VSHM takes about 1 s.

A break of Taylor series (4) results in neglecting the dispersion of photon scattering paths and, hence, in errors in brightness filed computations in the back hemisphere. To eliminate this disadvantage and obtain a complete solution of BP VRTE, let us find a non-small-angle (regular) addition to VSHM.

Regular solution part

As it was mentioned above, in this work BP is formulated only for a smooth small-angular addition, but not for scattered radiation as in the Chandrasekhar method. In this case, VSHM does not change the general form of BP VRTE:

$$\begin{cases} \mu\dfrac{\partial\vec{L}_R(\tau,\hat{l})}{\partial\tau}+\vec{L}_R(\tau,\hat{l})= \\ =\dfrac{\Lambda}{4\pi}\displaystyle\int_{4\pi}\ddot{R}(\hat{l}\times\hat{l}'\to\hat{z}\times\hat{l})\ddot{x}(\hat{l}\hat{l}')\ddot{R}(\hat{z}\times\hat{l}\to\hat{l}\times\hat{l}')\vec{L}_R(\tau,\hat{l}')d\hat{l}'+\vec{\Delta}(\tau,\hat{l}); \\ \vec{L}(0,\hat{l})\Big|_{\mu\ge 0}=\vec{0};\ \vec{L}(\tau_0,\hat{l})\Big|_{\mu\le 0}=-\vec{L}_{VSHM}(\tau_0,\hat{l}). \end{cases} \tag{6}$$

The source function is expressed on the base of VSHM $\vec{\Delta}(\tau,\hat{l})$:

$$\vec{\Delta}(\tau,\hat{l})=-\mu\frac{\partial}{\partial\tau}\vec{L}_{VSHM}(\tau,\hat{l},\hat{l}_0)+\vec{L}_{VSHM}(\tau,\hat{l},\hat{l}_0)+$$
$$+\frac{\Lambda}{4\pi}\int_{4\pi}\ddot{R}(\hat{l}\times\hat{l}'\to\hat{z}\times\hat{l})\ddot{x}(\hat{l}\hat{l}')\ddot{R}(\hat{z}\times\hat{l}\to\hat{l}\times\hat{l}')\vec{L}_{VSHM}(\tau,\hat{l}')d\hat{l}'. \tag{7}$$

Expansion of the scattering integral is one of the main problems both in scalar and vector radiation transfer theory, along with accounting for solution singularities. For this purpose, the complex circular CP basis[9] is used, diagonalizing the rotator matrix $\ddot{R}$ and allowing the scattering integral expansion. However, the use of a large-scale stabilizing solution[14] is impossible in case of complex numbers. Hence, after expansion of the scattering integral in the complex domain, the obtained vector coefficient system $\vec{f}_k^m(\tau)$ is transformed back to the real SP presentation (both for VSHM and for the smooth part (6) and solving source function (7) in BP VRTE). The vector discrete ordinates method (VDOM) with the boundary conditions in the Mark form (C. Mark) is chosen for solution (6) in view of its high computational efficiency in contrast to SHM. The BP VRTE has been solved in Ref. 11 by the Chandrasekhar method; this allows some designations from Ref. 11 to be used in this work. Finally, note that at tilted irradiation the references in Eqs. (4) and (6) differ. To align reference planes, we use their rotator-described rotation.

Consider the scattering integral in Eq. (6). To expand it, transform all the integrated matrices into the CP basis (transition matrix $\ddot{T}_{CS}$), diagonalizing the rotator. Let us use Eq. (2), the addition formula for GSF, and reverse matrix transition into the Stokes photometric basis (transition matrix $\ddot{T}_{CS}^{-1}=\ddot{T}_{SC}$):

$$\vec{I}_S=\ddot{T}_{SC}\frac{\Lambda}{4\pi}\times$$
$$\times\int_{4\pi}\ddot{T}_{CS}\ddot{R}(\chi)\ddot{T}_{SC}\ddot{T}_{CS}\ddot{x}(\hat{l}\hat{l}')\ddot{T}_{SC}\ddot{T}_{CS}\ddot{R}(\chi')\ddot{T}_{SC}\ddot{T}_{CS}\vec{L}(\tau,\hat{l}')d\hat{l}'=$$
$$=\frac{\Lambda}{4\pi}\int_{4\pi}\sum_{k=0}^{\infty}(2k+1)\sum_{m=-k}^{k}\exp(im(\varphi-\varphi'))\times$$
$$\times\ddot{P}_m^k(\mu)\ddot{\chi}_k\ddot{P}_m^k(\mu')\vec{L}(\tau,\hat{l}')d\hat{l}', \tag{8}$$

where complex polynomials are defined as a sum of the real $\ddot{P}_R(\mu)$ and imaginary $\ddot{P}_I(\mu)$ parts (each depends on k and m, but these indices are emitted here) as follows:

$$\ddot{P}_m^k(\mu)=\ddot{P}_R(\mu)+i\ddot{P}_I(\mu).$$

Grouping, as in scalar case, azimuth harmonics of the symmetric orders m and $-m$ in Eq. (8) and carrying out the above matrix transformation, we obtain

$$\vec{I}_S=\frac{\Lambda}{4\pi}\int_{4\pi}\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{k}(2-\delta_{0,m})\times$$
$$\times\left[\ddot{C}_m^k(\mu,\mu')\cos(m(\varphi-\varphi'))+\ddot{S}_m^k(\mu,\mu')\sin(m(\varphi-\varphi'))\right]\vec{L}(\tau,\hat{l}')d\hat{l}', \tag{9}$$

where

$$\ddot{C}_m^k(\mu,\mu')=\ddot{P}_R(\mu)\ddot{\chi}_k\ddot{P}_R(\mu')-\ddot{P}_I(\mu)\ddot{\chi}_k\ddot{P}_I(\mu');$$
$$\ddot{S}_m^k(\mu,\mu')=\ddot{P}_I(\mu)\ddot{\chi}_k\ddot{P}_R(\mu')+\ddot{P}_R(\mu)\ddot{\chi}_k\ddot{P}_I(\mu').$$

Similar to Ref. 11, it is convenient to introduce the following functions in Eq. (9): above-defined polynomials $\ddot{P}_k^m(\mu)$ in Eq. (5), selection matrices

$$\ddot{D}_1=\mathrm{diag}\{1,1,0,0\};\ \ddot{D}_2=\mathrm{diag}\{0,0,-1,-1\},$$

and phase matrices

$$\vec{\phi}_1(\varphi) = \mathrm{diag}\{\cos\varphi, \cos\varphi, \sin\varphi, \sin\varphi\};$$

$$\vec{\phi}_2(\varphi) = \mathrm{diag}\{-\sin\varphi, -\sin\varphi, \cos\varphi, \cos\varphi\},$$

which reduce Eq. (9) to the form

$$\vec{I}_S = \frac{\Lambda}{4\pi}\int_{4\pi}\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{k}(2-\delta_{0,m})\times$$

$$\times\left[\vec{\phi}_1(m(\varphi-\varphi'))\vec{A}_k^m\vec{D}_1 + \vec{\phi}_2(m(\varphi-\varphi'))\vec{A}_k^m\vec{D}_2\right]\vec{L}(\tau,\hat{\mathbf{l}}')d\hat{\mathbf{l}}', \quad (10)$$

where

$$\vec{A}_k^m = \vec{A}_k^m(\mu,\mu') = \vec{P}_k^m(\mu)\vec{\chi}_k\vec{P}_k^m(\mu').$$

According to the form of integral (10), the smooth part is conventionally presented by two azimuth-depending items

$$\vec{L}(\tau,\mu,\varphi) = \sum_{m=0}^{\infty}\left[\vec{\phi}_1(m\varphi)\vec{L}_1^m(\tau,\mu) + \vec{\phi}_2(m\varphi)\vec{L}_2^m(\tau,\mu)\right]. \quad (11)$$

Substituting Eq. (11) in Eq. (10), removing parentheses, and using orthogonality of the function $\vec{\phi}_{1,2}(\varphi)$, we obtain the equation for non-small-angle smooth addition (the scattering integral has the same expression in source function (7)) in the form ("~" means the regular part):

$$\vec{I}_S = \frac{\Lambda}{2}\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{k}\left[\vec{\phi}_1(m\varphi)\int_{-1}^{1}\vec{A}_k^m(\mu,\mu')\tilde{L}_1^m(\tau,\mu')d\mu' + \right.$$

$$\left. + \vec{\phi}_2(m\varphi)\int_{-1}^{1}\vec{A}_k^m(\mu,\mu')\tilde{L}_2^m(\tau,\mu')d\mu'\right], \quad (12)$$

where each item is defined from BP, similar to Eq. (6) (i = 1, 2):

$$\begin{cases} \mu\dfrac{\partial}{\partial\tau}\tilde{L}_i^m(\tau,\mu) + \tilde{L}_i^m(\tau,\mu) = \\ = \dfrac{\Lambda}{2}\displaystyle\sum_{k=0}^{\infty}(2k+1)\int_{-1}^{1}\vec{A}_k^m(\mu,\mu')\tilde{L}_i^m(\tau,\mu')d\mu' + \Delta_i(\tau,\mu), \quad (13) \\ \tilde{L}_i^m(0,\mu)\Big|_{\mu>0} = \vec{0};\ \tilde{L}_i^m(\tau_0,\mu)\Big|_{\mu<0} = -\vec{L}_{\mathrm{VSHM}}^m(\tau_0,\mu)\Big|_{\mu<0}, \end{cases}$$

provided the source function $\vec{\Delta}(\tau,\hat{\mathbf{l}})$ is defined. Note, that the function $\vec{\phi}_{1,2}(\varphi)$ was used in Ref. 11 owing to the block-diagonal form of the scattering matrix. The suggested method allows an arbitrary form of the scattering matrix in case of known matrix coefficients in its GSF-expansion.

To calculate the source function after aligning reference planes, the above-described techniques are used, i.e., transformation chain SP → CP → SP, GSF addition formula for scattering integral expansion in Eq. (7), and recurrent relations for the GSF system for vector coefficients.

On the base of the addition theorem, Eq. (4), and transition into the energy basis, the source function has the form

$$\vec{\Delta}(\tau,\hat{\mathbf{l}}) = \vec{T}_{SC}\left[-\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\exp(i2n\varphi)\times\right.$$

$$\left.\times\vec{Y}_k^{2n}(\nu)(\vec{\mathbf{1}} - \Lambda\vec{x}_k)\vec{\mathbf{l}}_k(\tau)\vec{f}_k^{2n}(0) - \mu\frac{\partial}{\partial\tau}\vec{L}_{\mathrm{VSHM}}(\tau,\hat{\mathbf{l}},\hat{\mathbf{l}}_0)\right].$$

The transformation equation for the differential term is easily obtained from Ref. (8); then the source function can be presented as

$$\vec{\Delta}(\tau,\hat{\mathbf{l}}) = \vec{T}_{SC}\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\exp(i2n\varphi)\vec{Y}_k^{2n}(\nu)\vec{F}_k^{2n}(\tau)\vec{f}_k^{2n}(0), \quad (14)$$

where the coefficients are defined by the following equation $\left(\vec{\mathbf{l}}_k = \exp(-(\vec{1} - \Lambda\vec{x}_k)\tau_0/\mu_0)\right)$:

$$\vec{F}_k^m(\tau) = \frac{1}{2k+1}\left[\vec{\mathbf{A}}_m^{k+1}(\vec{1} - \Lambda\vec{x}_{k+1})\vec{\mathbf{l}}_{k+1}(\tau) + \vec{\mathbf{B}}_m^k(\vec{1} - \Lambda\vec{x}_k)\vec{\mathbf{l}}_k(\tau) + \right.$$

$$\left. + \vec{\mathbf{A}}_m^k(\vec{1} - \Lambda\vec{x}_{k-1})\vec{\mathbf{l}}_{k-1}(\tau)\right] - (\vec{\mathbf{1}} - \Lambda\vec{x}_k)\vec{\mathbf{l}}_k(\tau).$$

The matrices in Eq. (14) have the form[8]:

$$\left[\vec{A}_m^k\right]_{rs} = \frac{1}{k}\sqrt{(k^2 - m^2)(k^2 - s^2)}\delta_{rs};$$

$$\left[\vec{B}_m^k\right]_{rs} = \frac{ms}{k(k+1)}(2k+1)\delta_{rs};\ r,s = +2,+0,-0,-2.$$

The reference functions differ in Eqs. (14) and (13) (the source function is defined with respect to the plane $\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0$ and VRTE – with respect to $\hat{\mathbf{l}}\times\hat{\mathbf{z}}$). It is necessary to use the rotator, presenting Eq. (14) as

$$\vec{\Delta}(\tau,\hat{\mathbf{l}}) = \vec{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0 \to \hat{\mathbf{l}}\times\hat{\mathbf{z}})\vec{T}_{SC}\times$$

$$\times\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\exp(i2n\varphi)\vec{Y}_k^{2n}(\nu)\vec{F}_k^{2n}(\tau)\vec{f}_k^{2n}(0)$$

or, writing additionally required transformation matrices of the basis:

$$\vec{\Delta}(\tau,\hat{\mathbf{l}}) = \vec{T}_{SC}\vec{T}_{CS}\vec{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0 \to \hat{\mathbf{l}}\times\hat{\mathbf{z}})\vec{T}_{SC}\times$$

$$\times\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\exp(i2n\varphi)\vec{Y}_k^{2n}(\nu)\vec{F}_k^{2n}(\tau)\vec{f}_k^{2n}(0) =$$

$$= \vec{T}_{SC}\vec{R}_{CP}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0 \to \hat{\mathbf{l}}\times\hat{\mathbf{z}})\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\exp(i2n\varphi)\times$$

$$\times\vec{Y}_k^{2n}(\nu)\vec{F}_k^{2n}(\tau)\vec{f}_k^{2n}(0).$$

Based on the addition theorem and reducing matrix to the real form, we have

$$\vec{\Delta}(\tau,\hat{\mathrm{l}},\hat{\mathrm{l}}_0)=\sum_{n=-1}^{1}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\sum_{q=-k}^{k}(-1)^q\ddot{\mathrm{T}}_{\mathrm{SC}}\ddot{\mathrm{Y}}_k^q(\mu)\ddot{\mathrm{T}}_{\mathrm{CS}}\ddot{\mathrm{T}}_{\mathrm{SC}}\times$$

$$\times\ddot{\mathrm{F}}_k^{2n}(\tau)\ddot{\mathrm{T}}_{\mathrm{CS}}\ddot{\mathrm{T}}_{\mathrm{SC}}\,\ddot{\mathrm{f}}_k^{2n}(0)\exp(iq(\varphi-\varphi_0))P_{2n,q}^k(\mu_0).$$

Finally, the source function has the following form:

$$\vec{\Delta}(\tau,\hat{\mathrm{l}},\hat{\mathrm{l}}_0)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\frac{2k+1}{4\pi}\ddot{\mathrm{P}}_k^m(\mu)\ddot{\Phi}_k(\tau)\times$$

$$\times\ddot{\mathrm{P}}_k^m(\mu_0)\vec{\mathrm{L}}_0\exp[im(\varphi-\varphi_0)], \tag{15}$$

where

$$\vec{\mathrm{L}}_0=[1\quad p\sin2\varphi_0\quad -p\cos2\varphi_0\quad q]^{\mathrm{T}}$$

is the SVP of the incident radiation with the linear polarization degree p and ellipticity q; φ_0 defines the azimuth position of the reference plane. In the SP basis, we have

$$\ddot{\Phi}_k(\tau)=\left\{\frac{1}{2k+1}\left[\ddot{\mathrm{A}}_{k+1}(\ddot{1}-\Lambda\ddot{\chi}_{k+1})\ddot{\mathrm{Z}}_{k+1}(\tau)\ddot{\mathrm{a}}_k+\right.\right.$$

$$\left.+4\frac{(2k+1)}{k(k+1)}(\ddot{1}-\Lambda\ddot{\chi}_k)\ddot{\mathrm{Z}}_k(\tau)\ddot{\mathrm{b}}+\ddot{\mathrm{A}}_k(\ddot{1}-\Lambda\ddot{\chi}_{k-1})\ddot{\mathrm{Z}}_{k-1}(\tau)\ddot{\mathrm{a}}_k\right]-$$

$$\left.-(\ddot{1}-\Lambda\ddot{\chi}_k)\ddot{\mathrm{Z}}_k(\tau)\right\},\qquad \ddot{\mathrm{Z}}_k=\exp\left(-(\ddot{1}-\Lambda\ddot{\chi}_k)\tau_0/\mu_0\right),$$

where

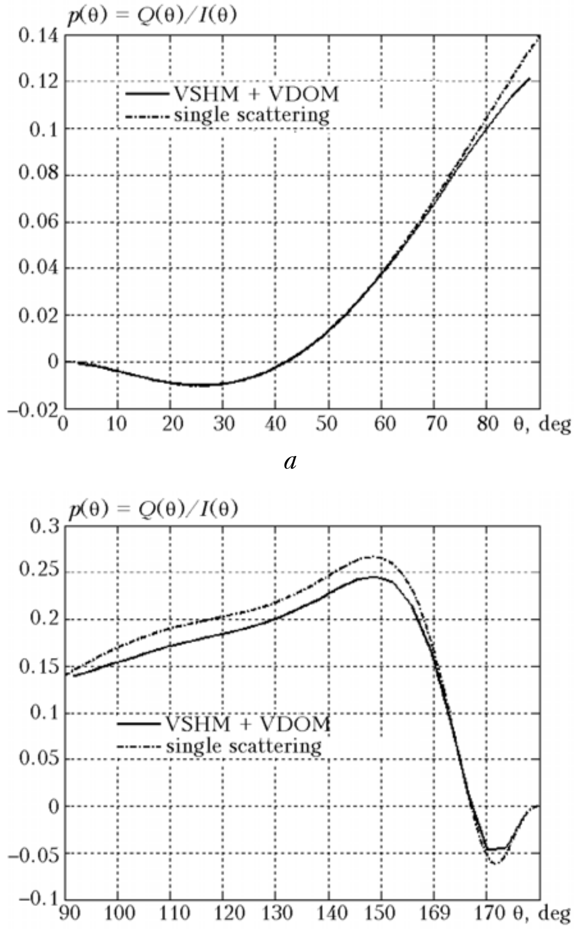
$$\ddot{\mathrm{a}}_k=\mathrm{diag}[k\quad \sqrt{k^2-4}\quad \sqrt{k^2-4}\quad k];$$

$$\ddot{\mathrm{A}}_k=\ddot{\mathrm{a}}_k/k;\ \ddot{\mathrm{b}}=\mathrm{diag}[0\quad 1\quad 1\quad 0].$$

The matrix coefficients, being substituted in Eq. (15) along with Eqs. (13), (4), and (1) (indices “VSHM” and “SA” are identical in this case), give the complete solution of BP VRTE for an arbitrary angle and polarization state. In this work, the problem has been solved by the matrix method of discrete ordinates with the boundary condition in the Mark form.

Some results

Below we give some computation examples of the complete BP VRTE solution by the above-described method. To simulate aerosol scattering, we used the Mie scattering by the Haze model L.[15] One of the effective ways of testing the solution is a comparison with single scattering approximation (Figs. 1 and 2) for weakly scattering media, described by simple analytical formulas. Figure 1 shows the comparison of results of polarization degree, computed by two methods for radiation, transmitted through the layer and diffusively reflected by it. Some downward deviation of the described VSHM + VDOM method from the single scattering approximation is evident.



b

Fig. 1. Angular dependence of the polarization degree $p(\theta)$: transmitted (a) and reflected (b) radiation.

This can be explained by the fact, that more than one scattering event is possible even in a thin layer at tilted paths, which results in radiation depolarization. The known event was also observed of complete depolarization of radiation, scattered forward and backscattered along the layer normal. The layer parameters are: $\tau = 0.1$ and $\Lambda = 0.9$.

Angular and depth dependences of the radiation polarization degree, calculated by the suggested method, are shown in Fig. 2. Computations confirm the known fact of radiation depolarization with an increase of scattering layer depth. Side “lobes” of the Mie scattering matrix are smoothed as a result of multiscattering, and monotone decrease of polarization degree depending on the angle without peaks (Fig. 2b) is observed for large depths ($\tau = 5$, 10). All computations show the known presence of neutral polarization points on the firmament.

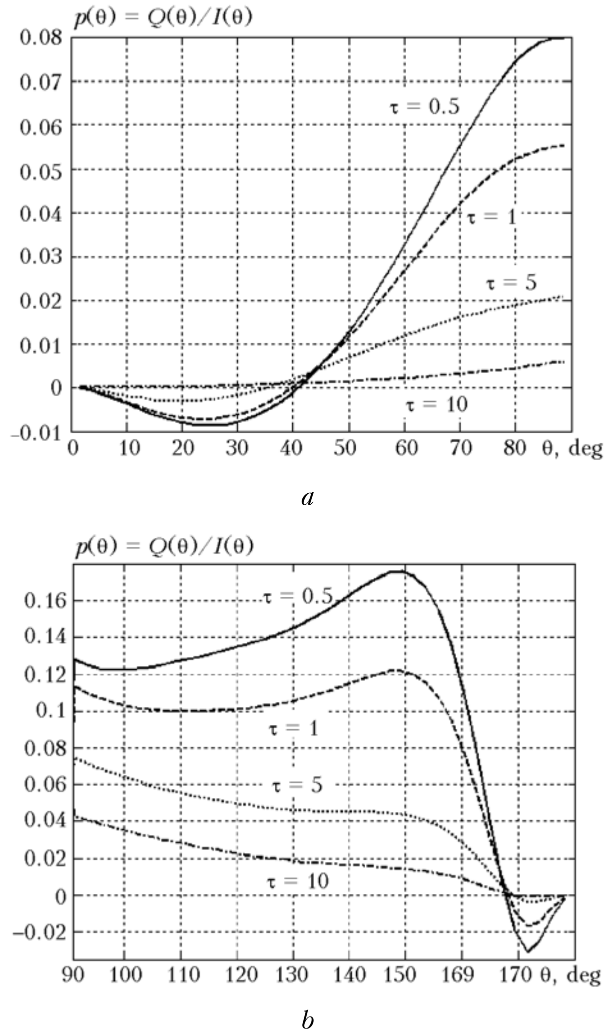


Fig. 2. Angular and thickness dependences of polarization degree $p(\theta)$: transmitted (*a*) and reflected (*b*) radiation.

Conclusion

In conclusion, emphasize an important moment: subtraction of only δ-singularity with further determination of scattered radiation seems to be ineffective in cases of strongly anisotropic scattering and math singularities in BP VRTE, described by the ray approximation. An effective way of solving the boundary problem in the transfer theory is its determination in the form of superposition of the whole anisotropic part and a smooth addition. In this case, the efficiency of the suggested method is quickly enhanced with an increase in scattering anisotropy, scattering medium stratification, as well as at transition to the three-dimensional geometry.

The solution in VSHM form, obtained on the base of VRTE simplification, is often sufficient for problems of sensing the descending radiation (radiation scattered to the forward hemisphere). Comparative simplicity of anisotropic part calculation, imposing the most exact requirements on the number of used harmonics, in VSHM along with addition smoothness determine the efficiency of the described method.

Note one more peculiarity of the polarized radiation transfer problems. In this work, the scattering integral is expanded in CP basis with further return into the real SP-presentation to apply matrix transformation [Ref. 14] and improve the solution stability. However, there is no need in the use of transformation chain SP–CP–SP, resulting in the transformation of form of matrix GSF (5). The use of $\vec{\mathrm{P}}_k^m(\mu)$ as the basis functions instead of $\ddot{\mathrm{Y}}_k^m(\mu)$ allows avoiding the going to the complex CP space, if the addition theorem, symmetry relations, and recurrent relations for $\vec{\mathrm{P}}_k^m(\mu)$ were proved before. This problem along with accounting for the complete scattering matrix instead of block-diagonal aerosol matrix and the going to three-dimensional geometry are objects of the current study for the authors.

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