A PROBABILISTIC APPROACH TO ACHIEVING DIFFRACTION-LIMITED OPTICAL SYSTEM UNDER RESOLUTION ATMOSPHERIC "SEEING" CONDITIONS

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We analyze the possibility of achieving diffract ion-limited resolution in a short-exposure atmosphere-telescope system for recording the image of a stationary isoplanatic object. We introduce the concept of the instantaneous spatial correlation length for atmospheric distortions of the radiation field, and using statistical computer simulation methods, we obtain the gamma distribution of its probability. Based on the use of this, distribution, we have studied the probability of good 'seeing' through the atmosphere. Using this approach, we have found the number of short time exposures required to give at least one with diffraction-limited resolution. These calculations were made for different ratios of the telescope diameter and Fried's parameter r_0. Some difficulties inherent in this probabilistic approach are discussed.

INTRODUCTION

The presence of atmospheric turbulence between an object and a telescope results in the actual resolution of the atmosphere-telescope system not being determined by the telescope diameter D, but by the statistical characteristics r_0 of atmospheric distortions of the light field; it is typically limited to one second of arc.

Such a resolution is inadequate for many important problems of observational astronomy, and defines the 'seeing' through a turbulent atmosphere. One of the classic techniques for solving the seeing problem is a probabilistic approach, where one passively waits for moments of good diffraction-limited seeing.

In actually, since atmospheric distortions of the light field are stochastic, there exists a probability that at certain moments the distortions at the telescope aperture are negligible. This may happen either the instantaneous variance $\sigma^2_{\theta,D}$ of the phase $\theta(\rho)$ distortions at the receiving aperture is smaller than unity[1], or the instantaneous value of the correlation length r_0 for distortions at such moments is larger than the diameter of the telescope. Obviously, images of an object obtained at such moments with short time exposures can have the the diffraction-limited resolution of the atmosphere-telescope system[2].

Below, we investigate the probability of such a situation, $P(\tilde{r}_0 \geq D)$, for different ratios between spatial characteristics of the atmospheric r_0 and the telescope diameter D. This will provide a way to estimate the mean number of short time exposures of the object image $K \sim 1/P$ required to obtain at least one image with diffraction-limited resolution.

In order to carry out these investigations it is necessary, firstly, to determine the statistical characteristics of the instantaneous spatial correlation length r_0 of atmospheric perturbations of the light field, and secondly, to estimate the probability that $\tilde{r}_0 \geq D$, yielding diffraction-limited resolution of the atmosphere-telescope system is obtained during short time exposures.

STATISTICAL CHARACTERISTICS OF THE r_0 VALUE

Let us first determine the instantaneous correlation length for atmospheric distortions of the optical field.

The mean correlation, length for these distortions[3], the so-called Fried's parameter r_0, is defined in terms of the optical transfer function (OTF) of the atmosphere-telescope system for long exposures[4] $\langle \tau^{\text{D-E}}_{\text{A-T}}(\rho, \lambda)\rangle$:

$$\int \langle \tau^{\text{D-E}}_{\text{A-T}}(\rho,\lambda) \rangle\, d\rho =$$

$$\int \exp\left\{-3.44\left[\frac{|\rho|}{r_0(\lambda,z^\circ)}\right]^{5/3}\right\} d\rho = \frac{\pi r_0^2(\lambda,z^\circ)}{4}, \quad (1)$$

which can be represented[5], as

$$r_0(\lambda,\ z^0) = r_0(\lambda_0,\ 0^0)(\lambda/\lambda_0)^{6/5}(\sec z^0)^{-3/5}, \quad (2)$$

where $r_0(\lambda_0,\ 0^0)$ is the mean correlation length for atmospheric distortions of the optical field occurring along a vertical path ($z^0 = 0^0$) at wavelength λ_0. This may be rewritten[3] as

$$r_0(\lambda_0,0^\circ)=\left[0.423\left(\frac{2\pi}{\lambda_0}\right)^2\int_0^L C_n^2(h)(h/L)^{5/3}dh\right]^{-3/5}, \quad (3)$$

where $C_n^2(h)$ is the vertical profile of the atmospheric refractive index, and L is the length of the turbulent path.

To simplify notation, we hereafter omit explicit functional dependences on λ_0 and z^0.

Thus, as in Eq. (1), one can determine the average value of the instantaneous spatial correlation length $\langle\tilde{r}_0\rangle$ for atmospheric distortions of the optical field by analyzing the behaviour of the mean resolution of the atmosphere-telescope system for short time exposures in terms of OTF for short-exposure images[4] $\langle\tau^{S\text{-}E}_{A\text{-}T}(\rho)\rangle$ as follows:

$$\int \langle \tau^{S-E}_{A-T}(\rho) \rangle \, d\rho = \int \exp\left\{-3.44\left[\frac{|\rho|}{r_0}\right]^{5/3}\left[1-\left(\frac{|\rho|}{D}\right)^{1/3}\right]\right\} d\rho = \frac{\pi\langle\tilde{r}_0\rangle^2}{4}. \quad (4)$$

It can easily be shown that $\langle\tilde{r}_0\rangle$ is related to the Fried parameter (2):

$$\langle\tilde{r}_0\rangle \simeq r_0[1 + 0.29(r_0/D)^{1/3}] \quad (5)$$

for $D \geq 3.5\ r_0$ (6)

It appears that $\langle\tilde{r}_0\rangle$ exceeds r_0 by 10 to 30 per cent for telescopes of moderate dimensions (D = 1 to 2 m), means that a corresponding improvement in resolution can be achieved by using short time exposures.

It is natural, in accordance with Eqs. (1) and (4) to define the instantaneous correlation length atmospheric distortions of the optical field $\tilde{r}_0$ in terms of the OTF (for short time exposures) for the atmosphere-telescope system:

$$\int |\tau^{S-E}_{A-T}(\rho)| \, d\rho = \frac{\pi\tilde{r}_0^2}{4}, \quad (7)$$

where the instantaneous OTF of the atmosphere-telescope system $\tau^{S\text{-}E}_{A\text{-}T}(\rho)$ determined in Ref. 6 can be represented in the form

$$\tau^{S-E}_{A-T}(\rho) = \frac{I}{S_A}\int W(\rho_1)W^*(\rho_1-\rho)\Delta(\rho_1)\Delta^*(\rho_1-\rho)d\rho_1. \quad (8)$$

where S_A is the receiving aperture of the telescope $S_A = \pi D^2/4$; $W(\rho)$ is the aperture function, define by

$$W(\rho) = A(\rho)\exp[j\theta_T(\rho)],$$

$A(\rho)$ is the pupil function, which has the value 1 inside the aperture and 0 outside it, $\theta_T(\rho)$ is function of telescope aberrations; $\Delta(\rho)$ is a function describing atmosphere-induced changes in the optical radiation, defined by

$$\Delta(\rho) = \exp\{j\theta_A(\rho)\}$$

when atmospheric amplitude fluctuations are negligible, and where $\theta_A(\rho)$ is a function of radiation phase fluctuations induced by the atmosphere. The latter function is a two-dimensional Gaussian process with zero average value $\langle\theta_A(\rho)\rangle = 0$ and a structure function of the form

$$D(\rho) = \langle[\theta(\rho_1)-\theta(\rho_2)]^2\rangle = 6.88[|\rho_1-\rho_2|/r_0]^{5/3}. \quad (9)$$

In order to find the probability distribution for $\tilde{r}_0$ (Eq. (7)) following the technique described in Ref. [7] we performed computer simulations of the instantaneous OTF for the atmosphere-telescope system. The OTF thus obtained were then used to calculate, according to Eq. (7), random values of $\tilde{r}_0$ for different ratios D/r_0. The length of each sequence $\tilde{r}_0$ values was 1200. Figure 1 illustrates the behaviour of the sample mean $\langle\tilde{r}_0\rangle$ (curve 1) and relative sample variance $\sigma^2_{\tilde{r}_0}/\langle\tilde{r}_0\rangle$ (curve 2) as functions of D/r_0. In this figure we also present analytic curves for $\langle\tilde{r}_0\rangle$ (curve 3) calculated using Eq. (6) and for r_0 (curve 4). Comparison of these curves shows good agreement between theory and experiment.

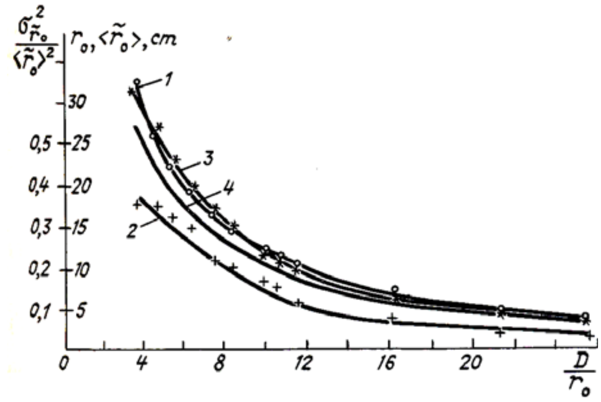

Fig. 1. Statistical characteristics of $\tilde{r}_0$: 1) mean value $\langle\tilde{r}_0\rangle$; 2) relative variance $\sigma^2_{\tilde{r}_0}/\langle\tilde{r}_0\rangle$; 3) $\langle\tilde{r}_0\rangle$ determined by Eq. (5); 4) Fried parameter r_0.

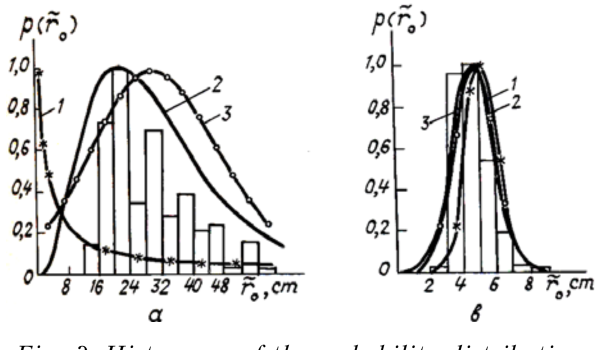

Fig. 2. Histograms of the probability distributions for $\tilde{r}_0$: a) D/r_0 = 3.75; b) D/r_0 = 21; 1) log-normal distribution; 2) gamma distribution; 3) normal distribution.

Figure 2 presents histograms of the probability distribution for $\tilde{r}_0$ obtained from statistical processing of the 1200 values synthesized for different ratios D/r_0.

Here we have also plotted analytic curves for three possible distribution laws for the probability of $\tilde{r}_0$, namely the log-normal law (curve 1)

$$p(\tilde{r}_0) = \frac{1}{(2\pi)^{1/2}\sigma_{\ln\tilde{r}_0}} \frac{1}{\tilde{r}_0} \exp\left\{-\frac{1}{2}\left[\frac{\ln\tilde{r}_0 - \langle\ln\tilde{r}_0\rangle}{\sigma_{\ln\tilde{r}_0}}\right]^2\right\}, \quad (10)$$

the gamma distribution (curve 2)

$$p(\tilde{r}_0) = \frac{\tilde{r}_0^{\alpha-1}\alpha^\alpha}{\Gamma(\alpha)\langle\tilde{r}_0\rangle^\alpha} \exp\left[-\frac{\alpha\tilde{r}_0}{\langle\tilde{r}_0\rangle}\right], \quad (11)$$

where $\alpha = \frac{\langle\tilde{r}_0\rangle^2}{\sigma^2_{\tilde{r}_0}}$ and $\Gamma(\alpha)$ is the gamma function, (12)

and the normal distribution (curve 3)

$$p(\tilde{r}_0) = \frac{1}{(2\pi)^{1/2}\sigma_{\tilde{r}_0}} \exp\left\{-\frac{1}{2}\left[\frac{\tilde{r}_0 - \langle\tilde{r}_0\rangle}{\sigma_{\tilde{r}_0}}\right]^2\right\}, \quad (13)$$

These distributions have been calculated for $\langle\tilde{r}_0\rangle$, $\langle\ln\tilde{r}_0\rangle$, and $\sigma^2_{\tilde{r}_0}$ values obtained from the experimental histograms.

A visual comparison of the theoretical distributions with the experimental histograms shows that in the range of D/r_0 values determined by (6), the gamma distribution (11) agrees with the experimental data best of all.

For a more accurate comparison of experimental data with the analytical distributions (10)–(13), we calculated the ratios of the moments $\langle\tilde{r}_0^m\rangle / \langle\tilde{r}_0\rangle^m$ for m = 2, 3, 4 and different ratios D/r_0. The data for these calculations are presented in Table 1. All three analytic distribution provide a good description of the probability distribution for $\tilde{r}_0$, by virtue of the central limit theorem, the natural normalization of distributions (10) and (11), and the fact that $\langle\tilde{r}_0\rangle$ tends[5] to r_0.

Note that this result is entirely consistent with the experimentally derived assumptions of the closeness of the probability distribution for r_0 to a log-normal distribution, which is an admissible approximation to the exact probability distribution for $\tilde{r}_0$ at large D/r_0.

Since moderate D/r_0 ratios are the most interesting in practice, it seems advisable, in investigations of the probabilistic approach to diffract ion-limited resolution of the atmosphere-telescope system, to use the gamma distribution for $\tilde{r}_0$ to assess the probability of good seeing conditions, as the gamma distribution best approximates the exact probability distribution of $\tilde{r}_0$ for any D/r_0.

Table 1.

$\langle\tilde{r}_0^m\rangle/\langle\tilde{r}_0\rangle^m$									
$D/r_0(\lambda, z^\circ)$	3.75			6.5			11.5		
m	2	3	4	2	3	4	2	3	4
1	1.4	2.81	7.24	1.23	2.1	5.34	1.096	1.32	1.74
2	1.4	2.51	5.49	1.23	1.81	3.08	1.096	1.31	1.68
3	1.28	1.88	3.03	1.22	1.69	2.54	1.095	1.29	1.6
4	3.8	19.1	109	3.15	13.4	64.9	2.39	7.93	19.1
$D/r_0(\lambda, z^\circ)$	16			21			25		
m	2	3	4	2	3	4	2	3	4
1	1.076	1.26	1.61	1.055	1.18	1.4	1.05	1.16	1.37
2	1.076	1.25	1.53	1.055	1.17	1.38	1.05	1.16	1.35
3	1.076	1.23	1.48	1.055	1.16	1.34	1.05	1.15	1.3
4	1.53	3.6	12.7	1.093	1.26	1.78	1.08	1.2	1.44

1 – experiment; 2 – gamma distribution; 3 – normal distribution; 4 – log-normal distribution.

THE PROBABILITY OF REACHING DIFFRACTION-LIMITED PERFORMANCE IN SHORT TIME EXPOSURES

In accordance with our chosen criterion for good seeing,

$$\tilde{r}_0 \geq D, \quad (14)$$

it seems natural to define the probability of good (diffraction-limited) seeing in terms of the gamma distribution (11) for $\tilde{r}_0$ as

$$P(\tilde{r}_0 \geq D) = \int_D^\infty p(\tilde{r}_0)d\tilde{r}_0. \quad (15)$$

It can easily be shown that the probability $P(\tilde{r}_0 \geq D)$ corresponds to the probability that the variance of atmospheric phase distortions at the receiving aperture (diameter D), determined in Ref. 9 to be

$$\sigma^2_{0,D} = 0.1411\ [D/r_0]^{5/3}, \quad (16)$$

must satisfy the inequality

$$\sigma^2_{0,D} \leq 0.141\ \text{rad}^2. \quad (17)$$

The condition (17) is much more stringent than that used by Fried[1],

$$\sigma^2_{0,D} \leq 1\ \text{rad}^2 \quad (18)$$

as a criterion for good seeing.

Table 2.

$D/r_0(\lambda, z^\circ)$	3.75	4.75	5.5	6.5
P	2.19×10^{-3}	7.02×10^{-4}	4.65×10^{-5}	5.53×10^{-7}

The probabilities of good seeing calculated using (15), and taking (11) into account for different D/r_0 ratios, are presented in Table 2 and in Fig. 3 (on a logarithmic scale). The linear dependence of $\ln P(\tilde{r}_0 \geq D)$ on the ratio $[D/r_0]^2$ indicates that it might be approximated by the empirical relationship

$$P(\tilde{r}_0 \geq D) \simeq k_1 \exp\left\{-k_2 \, [D/r_0]^2\right\}, D \geq 3.5\, r_0, \quad (19)$$

where the values of k_1 and k_2 determined from the data presented in Fig. 3 are

$$k_1 = 0.496,\ k_2 = 0.33. \quad (20)$$

It should be noted that such a negative exponential dependence of good seeing probability P on the ratio $(D/r_0)^2$ had been previously predicted by Hufnagel[7], and Fried confirmed it qualitatively[1] for the good seeing criterion (18). Fried's expression is of the form

$$P(\sigma^2_{\theta,D} \leq 1\ \text{rad}^2) \simeq k_1 \exp\left\{-k_2 [D/r_0]^2 [D/r_0]^2\right\},$$

$$D \geq 3.5 r_0, \quad (21)$$

where

$$k_1 = 5.6,\ k_2 = 0.1557.$$

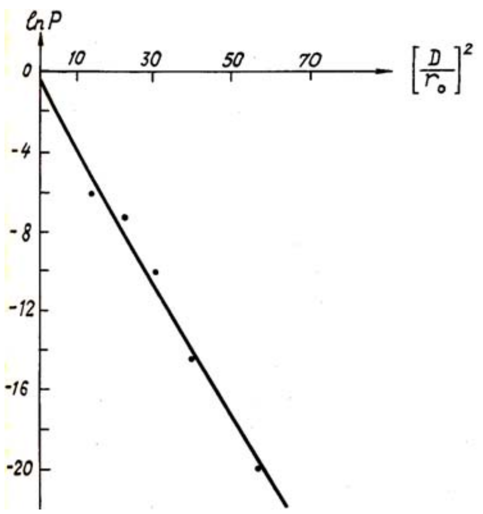

Fig. 3. The dependence of the logarithm of probability of good seeing on the ratio $(D/r_0)^2$.

A comparison of (19) and (21) shows that they differ only in the values of k_1 and k_2, the expression (21) obtained by Fried being more optimistic. This difference is both to the fundamentally different in the approaches used to derive the expressions (19) and (21) for probabilities of good seeing and to the difference in the criteria for good seeing (14) and (18), of which the latter (Fried's criterion) is the less stringent.

One can easily see from the data presented in Fig. 3 that in order to achieve diffraction-limited resolution at $D/r_0 = 3.5$, one should obtain about 100 short time exposures; the number of exposures at $D/r_0 = 4$ is no fewer than 400; at $D/r_0 = 4.5$ this number increases to 1600, and at $D/r_0 \geq 5$ the required number of exposures becomes so large (more than 7×10^3) that the foregoing probabilistic approach to diffraction-limited performance of the atmosphere-telescope system is scarcely practicable.

CONCLUSION

Our analysis of these results implies that in order to be able to achieve instantaneously diffraction-limited resolution in an atmosphere-telescope system, one must first of all accurately estimate $r_0(\lambda, 0^\circ)$ (Eq. (3)) at the telescope location, and use it to find $r_0(\lambda, z^\circ)$ (Eq. (2)) for given conditions of atmospheric seeing. It is also important to note that one should be able to reduce the telescope aperture (for example by using aperture stops) to values D_D which make the above probabilistic approach feasible, for example by ensuring that $D_D/r_0 = 3.5$. It is also worth noting that the diffraction-limited resolution discussed above assumes an ideal aberrationless ($\theta_T(\rho) = 0$) telescope of diameter D. Since any real telescope has aberrations, its effective diameter D_{eff} is always smaller than D. For that reason, achieving effective diffraction-limited resolution is more probable than achieving truly diffraction-limited resolution, which for a telescope with aberrations can hardly be achieved within the framework of the probabilistic approach. Moreover, the feasibility of this approach is also limited by the presence of uncompensated atmospheric dispersion, resulting in image coloration in the transverse direction, which at large zenith angles $z_0 > 45^\circ$ dominates the blurring of the image due to turbulence. These limitations clearly demonstrate the low efficacy of the classical probability approach to high angular resolution in systems of atmospheric seeing, underscoring the necessity of developing new nonclassical methods for pre-detection and post-detection cancellation of atmospheric distortions.

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