CALCULATION OF THE VARIANCE OF THE STRONG INTENSITY FLUCTUATIONS FOR LIGHT BEAMS PROPAGATING IN THE TURBULENT ATMOSPHERE

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Based on the representation of a complex wave field amplitude in the form of functional integral, an asymptotic formula is derived and calculations of the relative variance of strong intensity fluctuations for light beams propagated in the turbulent atmosphere is performed for arbitrary diffraction parameters of the aperture and spatial coherence of a source. Relative contribution of terms of an asymptotic series of different order to the variance is analyzed by way of varying the parameters of diffraction and the initial coherence of the source. It is established that the above–described effect of exceeding the unity by the level of saturation of the variance for strong turbulence on a path, when the structure function of phase on the inner scale of turbulence is greater than unity, is found to occur only in the case in which the beam is focused with large apertures and does not occur for other conditions of diffraction.

A number of papers is devoted to calculations of the intensity fluctuations of narrow optical beams in the turbulent atmosphere, in particular, see Refs. 1–6. However, the results obtained there either are based on approximate calculational methods[5,6] or yield the asymptotic estimates for the particular focusing and Fresnel's parameters of the transmitting aperture. In the present paper we study the variance of strong intensity fluctuations of optical beams propagated in the turbulent atmosphere for arbitrary diffraction parameters of the aperture and the coherence of the source.

For the calculation of the variance of the intensity let us make use of the representation of the complex wave field amplitude $U(x, \rho)$ in the form of a continuous integral[7,8]

$$U(x, \rho) = \frac{k}{2\pi i x}\int d^2\rho' U_0(\rho')\exp\left\{i\frac{k}{2x}(\rho-\rho')^2\right\}\times$$

$$\times\lim_{N\to\infty}\left(\frac{k}{2\pi i x}\right)^{N-1}\int d^2S_1\ldots d^2S_{N-1}\exp\left\{i\frac{k}{2x}\sum_{j=1}^{N-1}\mathbf{S}_j^2+\right.$$

$$\left.+\,i\frac{k}{2}\int\limits_x^0 dx'\varepsilon_1\left(x',\left(1-\frac{x'}{x}\right)\rho'+\frac{x'}{x}\rho+\sum_{j=1}^{N-1}\nu_j\frac{x'}{x}\mathbf{S}_j\right)\right\}, \qquad (1)$$

where x is the path length, ρ' and ρ are the two–dimensional vectors in the initial plane and the image plane, respectively, which are perpendicular to the direction of beam propagation ox', $k = 2\pi/\lambda$ is the wave number, $U_0(\rho)$ is the initial distribution of the field, $\varepsilon_1(x', t')$ is the fluctuating component of the dielectric constant of a medium, and $\nu_j(x'/x) = \sin(j\,\pi x'/x)\,(\sqrt{2}N\sin(j\,\pi/2N))$.(see Ref. 8)

The relative variance of the intensity $U(x, \rho) = U(x, \rho)U^*(x, \rho)$ on the beam axis

$$\sigma_I^2(x, 0) = \frac{<I^2>}{\ll I\gg^2} - 1 \qquad (2)$$

is expressed in terms of the average value $\ll I\gg = \ll U(x, 0)U^*(x, 0)\gg$ and of the second moment $\ll I^2\gg = \ll U(x, 0)U^*(x, 0)U(x, 0)\,U^*(x, 0)\gg$ of the intensity. The average intensity is calculated using the well–known methods[5,6,9] and currently it is not difficult. For the second moment in the case of a Gaussian beam using Eq. (1) after a number of transformations on the assumptions that the integral of the field $\varepsilon_1(x', t)$ along the path in Eq. (1) is the normally random value and the field is locally homogeneous, isotropic, and δ–correlated, we derive

$$\ll I^2(x, 0)\gg = \left(\frac{1}{2\pi}\right)^{2(N-1)}\Omega^4\frac{1}{\pi^3}\int d^2S_{1-3}\times$$

$$\times\exp\left\{-g^2\mathbf{S}_1^2-\mathbf{S}_2^2-\mathbf{S}_3^2+i\,2\Omega\left(1-\frac{x}{F}\right)\mathbf{S}_2\mathbf{S}_3\right\}\times$$

$$\times\int d^2S'_1d^2S''_1\ldots d^2S'_{N-1}d^2S''_{N-1}\exp\left\{i\sum_{j=1}^{N-1}\mathbf{S}'_j\,\mathbf{S}''_j\right\}\times$$

$$\times\exp\left\{-\frac{\pi k^2}{4}x\left(2\frac{x}{k}\right)^{5/6}\Omega^{5/6}\int\limits_0^1 d\xi\left[H\left((1-\xi)(\mathbf{S}_1+\mathbf{S}_2)+\right.\right.\right.$$

$$\left.+(2\Omega)^{-1/2}\sum_{j=1}^{N-1}\nu_j(\xi)\mathbf{S}''_j\right)-H\left((1-\xi)(\mathbf{S}_2+\mathbf{S}_3)+\right.$$

$$\left.+(2\Omega)^{-2}\sum_{j=1}^{N-1}\nu_j(\xi)(\mathbf{S}'_j+\mathbf{S}''_j)\right)+H\left((1-\xi)(\mathbf{S}_1+\mathbf{S}_3)+\right.$$

$$\left.+(2\Omega)^{-1/2}\sum_{j=1}^{N-1}\nu_j(\xi)\mathbf{S}'_j\right)+H\left((1-\xi)(\mathbf{S}_3-\mathbf{S}_1)+\right.$$

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$$+ (2\Omega)^{-1/2}\sum_{j=1}^{N-1} \nu_j(\xi)\mathbf{S}'_j\Big) - \Big((1-\xi)(\mathbf{S}_3 - \mathbf{S}_2) +$$

$$+ (2\Omega)^{-1/2}\sum_{j=1}^{N-1} \nu_j(\xi)(\mathbf{S}'_j - \mathbf{S}''_j)\Big) + H\Big((1-\xi)(\mathbf{S}_1 - \mathbf{S}_2) -$$

$$- (2\Omega)^{-1/2}\sum_{j=1}^{N-1} \nu_j(\xi)\mathbf{S}''_j\Big)\Big]\Big\}, \quad (3)$$

where $\Omega = \frac{ka^2}{x}$, $g^2 = 1 + \Omega^2\left(1 - \frac{x}{F}\right)^2$, a is the effective beam radius, F is the curvature radius of the beam wavefront in the initial plane, $H(\mathbf{t}) = 2\int d^2\mathbf{k}\Phi_\varepsilon(\mathbf{k})[1 - \cos\mathbf{kt}]$, and $\Phi_\varepsilon(\boldsymbol{\kappa})$ is the spectral density of fluctuations of the dielectric constant of air.

In the case of radiation focusing $\left(\frac{x}{F} = 1\right)$ with large apertures ($\Omega \to \infty$) an equation follows from Eq. (3) which coincides with the corresponding representation of $\langle\langle I^2 \rangle\rangle$ in the phase approximation of the Huygens–Kirchhoff method employed for the analysis of this particular case in Ref. 1.

For the Kolmogorov spectrum of atmospheric turbulence[9,10] the last exponent in Eq. (3) denoted by A can be represented in the form

$$A = \exp\{B\}\exp\{D\} = \exp\Big\{-0.3\beta_0^2\Omega^{5/6}\int_0^1 dx\int d^2\mathbf{k}|\boldsymbol{\kappa}|^{-11/3} \times$$

$$\times \sum_{l=0,1} |(1-\xi)(\mathbf{S}_1 + (-1)^l\mathbf{S}_2) + (-1)^l(2\Omega)^{-1/2}\sum_{j=1}^{N-1}\nu_j(\xi)\mathbf{S}''_j|^{5/3}\Big\} \times$$

$$\times \exp\Big\{-0.3\beta_0^2\Omega^{5/6}\int_0^1 d\xi\int d^2\mathbf{k}|\boldsymbol{\kappa}|^{-11/3} \times$$

$$\times\Big[\sum_{l=0,1} |(1-\xi)(\mathbf{S}_3 + (-1)^l\mathbf{S}_1) + (2\Omega)^{-1/2}\textbf{Ошибка!}\,|^{5/3} -$$

$$- |(1-\xi)(\mathbf{S}_3 + (-1)^l\mathbf{S}_2) + (2\Omega)^{-1/2} \times$$

$$\times \sum_{l=0,1}(\nu_j(\xi)(\mathbf{S}'_j + (-1)^l(\mathbf{S}''_j|^{5/3}\Big]\Big\}, \quad (4)$$

where $\beta_0^2 = 1.23C_n^2k^{7/6}x^{11/6}$ is the variance of the intensity fluctuations of a plane wave, calculated to the first order of the smooth perturbation method. The quantity β_0^2 can be conveniently used as a parameter characterizing the turbulent conditions of propagation.[9,10] To derive the relation for $\sigma_I^2(x, 0)$ for strong intensity fluctuations when the parameter β_I^2 is much greater than unity, me use the method of asymptotic (as $\beta_0^2 \to \infty$) calculation of the integrals of type (3).[10,5,6,14] In accordance with this method, the term A of the integrand in Eq. (3) can be represented by a series in the domain important for integrating

$$A = 2\exp\{B\}\Big[1 - D + \frac{1}{2}D^2 + \ldots\Big], \quad (5)$$

and then it can be integrated over all vector variables taking into account the first three terms .

It should be noted that when analyzing the strong intensity fluctuations of collimated narrow optical beams, the authors of Refs. 2–4 took into account only the terms analogous to the first two terms enclosed in square brackets in Eq. (5) when they calculated an asymptotic series, while in Refs. 1 and 2 the second term was replaced by the term analogous to the third term of Eq. (5) and was dominant in the asymptotic representation of σ_I^2 for the case of radiation focusing with large apertures $\left(\frac{x}{F} = 1, \Omega \to \infty\right)$.

Substituting Eq. (5) into Eq. (3), we derive after integrating over all vector variables

$$\sigma_I^2(x, 0) = 1 + \sigma_{I,\mathrm{c}}^2 + \sigma_{I,\mathrm{F}}^2, \quad (6)$$

where

$$\sigma_{I,\mathrm{c}}^2 = \pi^2\frac{0.033}{0.31}\beta_0^2\,\Gamma(7/6)\frac{2^{7/6}}{g_r^2}\int_0^1 d\xi(1-\xi)^{1/3}\Big[\Omega\Big(1 + \frac{1}{g_r^2} - \frac{\Omega^2 G}{g_r^2} +$$

$$+ \xi\left(\frac{1}{2}D_s\right)^{6/5}/\Omega^2\Big]^{-7/6}(\xi g_r^2 + (1-\xi)\,\Omega^2 G)^2, \quad (7)$$

$$\sigma_{I,\mathrm{F}}^2 = \int_0^1 d\xi'\left[\int_0^{\xi'} d\xi''\,F(g_r, \gamma_1(g_r)) + \int_{\xi'}^1 d\xi''\,F(g_r, \gamma_1(g_r))\right], \quad (8)$$

$$F(g_r, \gamma_i(g_r)) = 1.12\cdot10^{-2}\frac{\pi^2\Gamma^2(7/6)}{2^{1/3}}\frac{D_s^2}{g_r^4}(1-\xi')^{5/3}(1-\xi'')^{5/3} \times$$

$$\times\left[\frac{36}{\gamma(\xi')\gamma(\xi'')^{1/6}}\,{}_2F_1\left(1/6;\,1/6;\,1;\,\frac{\gamma_i^2(g_r)}{\gamma(\xi')\gamma(\xi'')}\right) +\right.$$

$$\left.+\frac{1}{2}\frac{\gamma_i^2(g_r)}{\gamma(\xi')\gamma(\xi'')^{1/6}}\,{}_2F_1\left(7/6;\,7/6;\,3;\,\frac{\gamma_i^2(g_r)}{\gamma(\xi')\gamma(\xi'')}\right)\right], \quad (9)$$

$$g_r^2 = g^2 + \left(\frac{1}{2}D_s\right)^{6/5},\; D_s = 2.84\,\beta_0^2\,\Omega^2,$$

$$G = \left(1 - \frac{x}{F}\right) - \xi(1 - \xi/2)\,(1/2D_s)^{6/5},$$

$$\gamma(\xi) = 1 + \frac{8}{3}\alpha^2(\xi) - q^2(\xi)/g_r^2,$$

$$\gamma_1 = 1 + \frac{4}{3}\alpha(\xi')\alpha(\xi'')(\xi'(1-\xi''))^{-1}(\xi'(2-\xi') - (\xi'')^2 - q(\xi')q(\xi'')/g_r^2,$$

$$\gamma_2 = 1 + \frac{4}{3}\alpha(\xi')\alpha(\xi'')(\xi''(1-\xi'))^{-1}(\xi''(2-\xi'') - (\xi')^2 - q(\xi')q(\xi'')/g_r^2,$$

$$\alpha(\xi) = 0.76\xi\beta_0^{6/5}\Omega^{1/2},\; q(\xi) = \Omega\left(1 - \frac{x}{F}\right) - \mu(\xi),$$

$\mu(\xi) = 0.76\,\xi(2 - \xi)\,\beta_0^{12/5}$,

${}_2F_1(x, y, z, t)$ is the hypergeometric Gauss function and $\Gamma(x)$ is the gamma function.

Formula (6) can be used to calculate the relative variance of the strong intensity fluctuations for arbitrary parameters $\frac{x}{F}$ and Ω. In the case of collimated beams $\left(\frac{x}{F} = 0\right)$ and beams focused with the apertures whose Fresnel's number $\Omega \lesssim \beta_0^{12/5}$ we have $\sigma_{I,c}^2 \sim \beta_0^{-4/5}$ and $\sigma_F^2 \sim \beta_0^{-8/5}$. In the case of focusing with large apertures $\Omega \gg \beta_0^{12/5}$) we derive $\sigma_{I,c}^2 \sim \frac{\beta_0^2}{\Omega^{7/6}}$ and $\sigma_{I,F}^2 \sim D_s^{-2/5}$ that agrees with the results of Refs. 1–4.

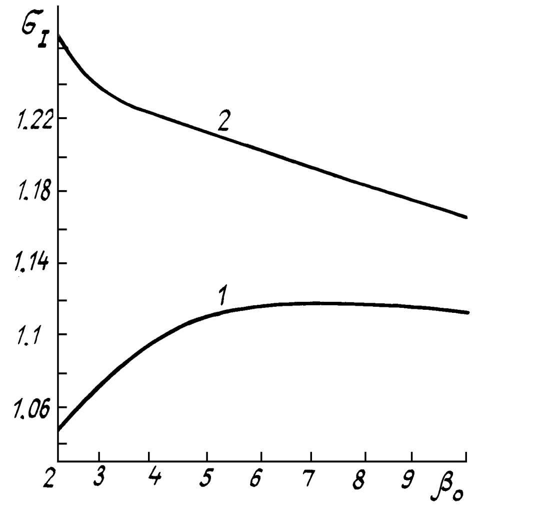

FIG. 1. Dependence of the relative variance of the infensity of a focused beam ($x/F = 1$ and $\Omega = 25$) on the parameter β_0^2: 1) calculations neglecting the term $\sigma_{I,F}^2$ ($\sigma_I^2 = 1 + \sigma_{I,c}^2$) and 2) calculations according to formula (6).

Figure 1 shows the change of the relative contribution of the terms $\sigma_{I,c}^2$ and $\sigma_{I,F}^2$ to the variance of the collimated beam as a function of the intensity of atmospheric turbulence along the path. It can be seen that for $\beta_0^{12/5} \le \Omega$ the main contribution to σ_I^2 exceeding unity comes from the term $\sigma_{I,F}^2$. With increase of β_0^2, the contribution of $\sigma_{I,F}^2$ decreases and for $\beta_0^{12/5} > \Omega$ the variance behavior is primarily determined by the first two terms of expansion (5).

Figure 2 shows the results of calculation of σ_I^2 on the axis of the collimated beam as a function of the diffraction parameter Ω. Experimental data are also plotted here. It follows from the figure that taking into account the additional, in comparison with the results of Refs. 3 and 4, term $\sigma_{I,F}^2$ of higher order enables us to obtain a good qualitative agreement with the experimental dependence without averaging over the positions of the observation point relative to the beam axis.[4] Simultaneous account of the term $\sigma_{I,F}^2$ and averaging over the positions of the observation point[4] (curve *2*) leads to a better agreement between theoretical and experimental dependences than in the absence of $\sigma_{I,F}^2$ (curve *2'*).

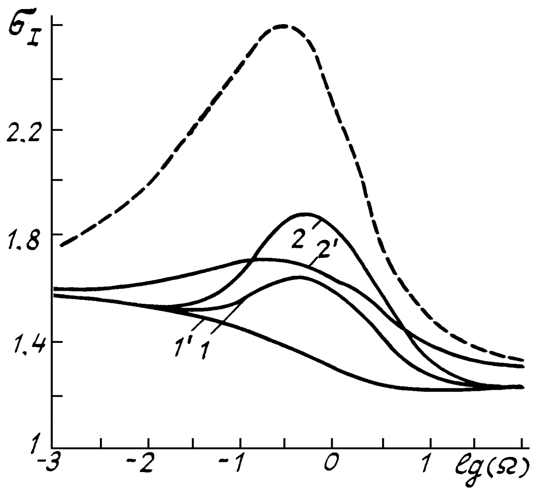

FIG. 2. Dependence of $\sigma_I^2(\Omega)$: 1) calculations based on the formula (6) for $\beta_0 = 1.8$, 1') calculations based on the formula $\sigma_I^2 = 1 + \sigma_{I,c}^2$ (Ref. 4) for the same value of β_0, 2 and 2') calculations with averaging over the positions of the observation point. Dashed curve refers to the experimental data.

Let us consider in detail the case of the radiation focusing with large apertures ($\Omega \gg \beta_0^{12/5}$) when the term $\sigma_{I,c}^2$ is negligible. Let us assume, that the turbulent distortions of the wave are so great that the parameter $\rho_c \sim \sqrt{x/k}\,\beta_0^{6/5}$ (Ref. 6) becomes smaller than the inner scale of the turbulence: $\rho_c < l_0$. Then the index 5/3 in the exponent B in Eqs. (4) and (5) is replaced by 2 and the dependence on l_0 appears.[9,11] After integrating as a result we derive for σ_I^2 the formula[11,5,6]

$$\sigma_I^2 = 1 + 2.5 l_a^{2/3}[1 + 1.6 D_s^{-1}(l_0)] + 4.7 l_a^{1/3}[1 + 4 D_s^{-1}(l_0)] + 0(l_a^2),$$
$$l_a < 1,\ D_s(l_0) \gg 1, \qquad (10)$$

where $D_s(l_0) = 1.1 C_n^2 k^2 x \left(\frac{9}{8}\right)^3 l_0$ and $l_a = \left(\frac{9}{8}\right)^3 \frac{l_0}{2a}$ demonstrating that the saturation level of the variance exceeds unity by the value determined by the ratio of the inner scale of the turbulence to the aperture radius. For the collimated beams when the dominant term of asymptotic expansion is the term $\sigma_{I,c}^2$ the condition $\rho_c < l_0$ leads only to the substitution of the term $\beta_0^{-1/3}$ for $\beta_0^{-4/5}$, and, according to this power–law function, the saturation of σ_I^2 at unity occurs. In Ref. 11 relation (10) was illegitimately expanded on the collimated beams on the basis of the calculations in the phase approximation of the Huygens–Kirchhoff method which was inconsistent with the conclusions of Refs. 3 and 4 and gave rise to discussions.

Measurements of the variance of the intensity of the focused beam with the use of the laboratory setup under conditions in which the parameter D_s approached very large values up to 500 were presented in Ref. 12 . The stable saturation of σ_I^2 observed in Ref. 12 at the level of $\sigma_I^2 = 1.06$ for $120 < D_s < 500$ in some extent can be explained by dependence (10).

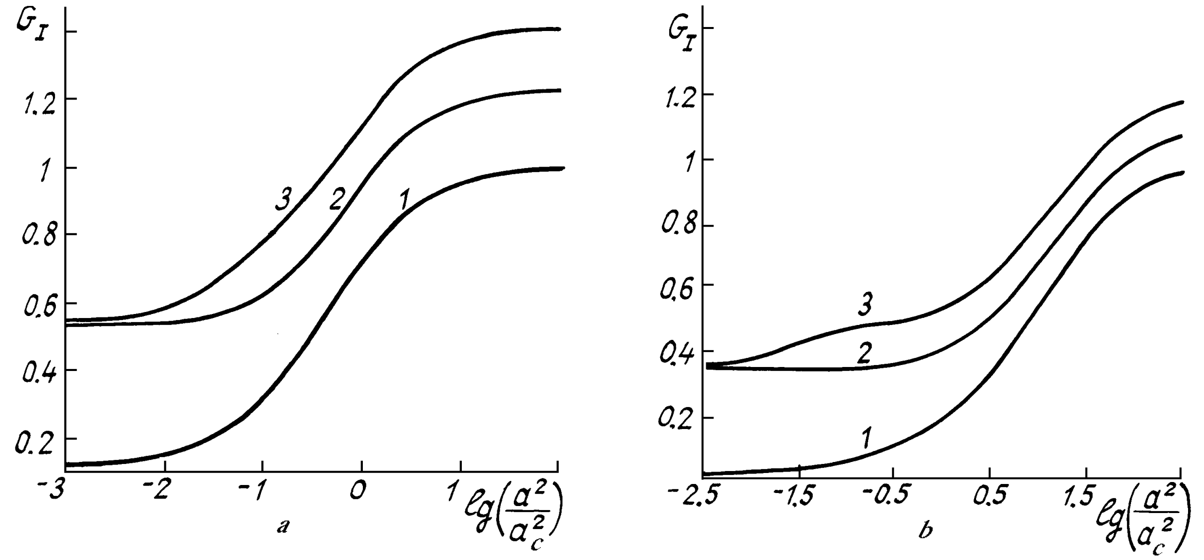

FIG. 3. Calculation of $\sigma_I^2\left(\frac{a^2}{a_c^2}\right)$ *(from formula (11): a) collimated beam ($x/F = 0$ and $\Omega = 1$), b) focused beam ($x/F = 1$, $\Omega = 25$), curves 1, 2, and 3 are calculated taking into account the first, the first two, and all three terms in Eq. (11), respectively.*

Formula (6) can be generalized for the case of a partially coherent radiation when the initial field $U_0(\rho')$ is random. Let us assume that the time constant of coherence τ_c of the source is much less than the time constant of the receiver τ_r. Then the fourth moment of the initial field separates in the product of the second moments[6] with the spatial coherence radius a_c.[6,9] After the calculations analogous to the foregoing instead of Eq. (6) we derive the formula

$$\sigma_I^2(x,\,0) = \frac{\tilde{g}_r^2}{\tilde{g}_r^2 + r^2} + \tilde{\sigma}_{I,c}^2 + \tilde{\sigma}_{I,F}^2, \qquad (11)$$

where

$$\tilde{g}_r^2 = g_r^2 + \frac{a^2}{a_c^2},\quad r^2 = 1.52\,\beta_0^{12/5}\,\Omega\,\frac{a^2}{a_c^2},$$

$\tilde{\sigma}_{I,c}^2$ is the term analogous to $\sigma_{I,c}^2$ in Eq. (6) but containing the additional parameter and having more complicated form,[13,6]

$$\tilde{\sigma}_{I,F}^2 = \frac{1}{2}\int\limits_0^1 d\xi'\left\{\int\limits_0^{\xi'} d\xi''\left[F(\tilde{g}_r,\,\gamma_1(\tilde{g}_r)) + C(\gamma_1)\right] + \right.$$

$$\left. + \int\limits_{\xi'}^1 d\xi''\left[F(\tilde{g}_r,\,\gamma_2(\tilde{g}_r)) + C(\gamma_2)\right]\right\} \qquad (12)$$

the functions $C(\gamma_i)$ have the same form as $F(\tilde{g}_r,\,\gamma_i(\tilde{g}_r))$ in Eq. (9) with the only difference that $\tilde{g}_r^2$ should be replaced by $\tilde{g}_r^2 + r^2$, while α' and α'' and μ' and μ'' are multiplied by $\left(1 + \frac{a^2}{a_c^2}\right)^{1/2}$ and $\left(1 + \frac{a^2}{a_c^2}\right)$, respectively. In addition, the first term enclosed in the square brackets in Eq. (9) for $C(\gamma_i)$ has the multiplier $G(p + s - t)$ and the multiplier $G(2p + s)$ before the second term, where

$$G = (1 + a^2/a_c^2)^{1/3}/(\tilde{g}_r^2(\tilde{g}_r^2 + r^2)),\quad p = \tilde{g}_r^2 + \tilde{g}_r^2 r^2 + \frac{a^2}{a_c^2}\tilde{g}_r^4,$$

$$s = r^4 + \left(\frac{a^2}{a_c^2}\right)\tilde{g}_r^4,\quad t = \frac{a^2}{a_c^2}\tilde{g}_r^2 r^2.$$

It follows from Eq. (11) that when the spatial coherence of the initial field degrades, the saturation level for the relative variance becomes less than unity.[13,6] Figure 3 shows the ralative contributions of three components in the relative variance in Eq. (11) as functions of the source coherence. Thus, formula (11) summarizes separate results given in Refs. 1–4 and 13 and can be used to calculate the variance of the strong intensity fluctuations for arbitrary parameters of focusing, diffraction, and initial spatial coherence of radiation.

REFERENCES

1. K.S. Gochelashvili, Kvant. Elektron. **1**, No. 4, 848 (1974).
2. M.I. Charnotskii, in: *Abstracts of Reports at the 5th All–Union Symposium on the Propagation of Laser Radiation in the Atmosphere,* Tomsk (1979), Vol. 1, p. 74.
3. I.G. Yakushkin, Izv. Vyssh. Uchebn. Zaved. Ser. Radiofiz. **19**, No. 3, 384 (1976).

4. S.I. Belousov and I.G. Yakushkin, Kvant. Elektron. **7**, No. 3, 530 (1980).
5. V.L. Mironov, *Laser Beam Propagation in the Turbulent Atmosphere* (Nauka, Novosibirsk, 1981), 246 pp.
6. V.E. Zuev, V.A. Banakh, and V.V. Pokasov, *Optics of the Turbulent Atmosphere* (Gidrometeoizdat, Leningrad, 1988), 270 pp.
7. V.I. Klyatskin, *Statistical Description of Dynamic Systems with Fluctuating Parameters* (Nauka, Moscow, 1975), 239 pp.
8. V.I. Tatarskii and V.U. Zavorotnyi, Proc. SPIE 6**42, 27**6, (1986).
9. A.S. Gurvich, A.I. Kon, V.L. Mironov, and S.S. Khmelevtsov, *Laser Radiation in the Turbulent Atmosphere* (Nauka, Moscow, 1976), 280 pp.
10. S.M. Rytov, Yu.A. Kravtzov, and V.I. Tatarskii, *Introduction in the Statistic Radio* Physics, Vol. 2. Random Fields (Nauka, Moscow, 1978), 464 pp.
11. V.A. Banakh and V.L. Mironov, Kvant. Elektron. **5**, No. 7, 1535 (1978).
12. B.S. Agrovskii, A.S. Gurvich, and M.A. Kallistratova, Izv. Vyssh. Uchebn. Zaved, Ser. Radiofiz. **21**, No. 2, 212 (1978).
13. V.A. Banakh, V.M. Buldakov, and V.L. Mironov, Opt. Spektrosk. **54**, No. 6, 1054 (1983).
14. M.I. Charnotskii, in: *Abstracts of Reports at the 11th All–Union Symposium7 on the Propagation of Laser Radiation in the Atmosphere and Water Media,* Tomsk (1991), p. 5.