Statistics of photocounts produced by optical signals with lognormal intensity distribution

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Statistical characteristics of the fluxes of single-electron pulses (SEP) and photocounts (PC) were studied using a numerical simulation method. The radiation recorded is assumed to have lognormal intensity distribution. The numerical simulation algorithm is described, which is realized following the "intensity – photodetector – SEP flux – PC flux" scheme. The results calculated for different statistical characteristics of the SEP and PC fluxes are discussed and interpreted. A significant difference between them has been revealed for the lognormal and exponential intensity distributions.

The study of statistical characteristics of photocurrent is necessary for solving wide range of problems related with statistical modeling of signals and noises aimed at optimization of different optical systems, estimation of the efficiency of these systems, their modeling, in processing and interpretation of the measurement results. A typical system intended for recording weak radiation fluxes in the visible range comprises a photomultiplier tube (PMT), a pulse-height discriminator, and a pulse shaper that sends out uniform pulses to a counter. The flux of single-electron pulses (SEP) at the PMT output corresponds to the flux of photoelectrons (PE) emitted from the cathode, and photocurrent is the groups of partially overlapping SEPs. A part of pulses from the PTM is not recorded by the counter because of the inertia or the "dead" time (DT) of different elements of the photon-counting system. As a result, the flux of recorded SEPs or photocounts (PC) is formed.

The difference between statistical characteristics of the SEP and PC fluxes increases with the increasing intensity of the radiation recorded. This circumstance is most essential in recording optical signals that have a wide dynamical range, for example, in laser sounding of the atmosphere and optical ranging. In this case it is necessary to record and process them in different modes of detection and to take into account the distorting effects due to DT, which is one of the main sources of measurement errors. Definitions of the modes of recording optical signals as photon counting, intermediate, current (analog) can be found in Ref. 1. Figure 1 illustrates the schematic of a photon counting system. One can see from Fig. 1 that it records only the second SEP and the group of two overlapping SEPs whose heights exceed the discrimination threshold level, i.e. the number of SEP $n = 4$, while the number of PCs $m = 2$. The first SEP has not been recorded, because its amplitude is lower than the U_d level.

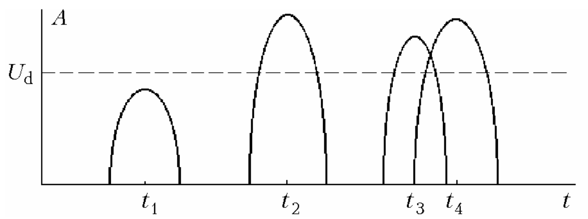

Fig. 1. Example of realization of the SEP sequence: $n = 4$, $m = 2$.

Here and below the PMT with a counter are considered as a paralyzable recording system. This means that SEP is recorded only in the case when time interval between its occurrence and the occurrence of most recent previous pulse, even not recorded, is less than the width of a SEP, $\Delta\tau$, at the level of the discrimination threshold U_d. Classification of different types of photon counting systems can be found in Ref. 2.

Transformation of the statistical characteristics of the SEP flux by counters of different types was considered by many authors (see, for example, Refs. 2–4). To carry out such a research, it is necessary to know the joint-probability density $f(t_1, t_2, \ldots, t_n)$ of the time moments $\{t_i\}$ of the occurrence of PE or an SEP, which for a double stochastic Poisson flux is defined by the formula[5]:

$$f(t_1, t_2, \ldots, t_n) = \left\langle \prod_{i=1}^{n} \lambda(t_i) \exp[-\int_0^T \lambda(t) dt] \right\rangle_\lambda , \quad (1)$$

where $<\ldots>_\lambda$ means averaging over realization of the optical radiation power

$$\lambda(t) = \eta \iint_{S^*} I(\mathbf{r}, t) d^2\mathbf{r}, \quad (2)$$

collected with a receiving aperture, S^* ($I(\mathbf{r}, t) = E(\mathbf{r}, t)E^*(\mathbf{r}, t)$ is the intensity of the

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optical radiation field at the point **r** at the time moment t; $E(\mathbf{r}, t)$ is the field strength, η is the quantum efficiency of the photodetector).

In the general case, the double stochastic Poisson flux of SEP generated by random power of the recorded radiation is not the reconstruction process, because the following condition is not fulfilled for it:

$$f(t_1,t_2,\ldots,t_n) = f_1(\tau_1)\prod_{i=1}^{n} f_2(\tau_i), \qquad (3)$$

that is seen from comparison of the formulas (1) and (3) (in Eq. (3) $\tau_i = t_i - t_{i-1}$ are the time intervals between the neighbor SEPs). So direct use of the known results for determining the statistical characteristics of the PC flux generated by a double stochastic SEP flux is impossible. The exception is the case of SEP and PC fluxes generated by the detected radiation with the time correlation radius τ_c essentially exceeding the time of sampling T.[6–8] Statistical characteristics of the PC flux of radiation with an exponential intensity distribution or the Gaussian field $E(\mathbf{r}, t)$ in the plane of the receiving aperture have been considered in Refs. 9 to 11. The method of numerical simulation of the optical detector operation in the photon-counting mode was used.

The random field at the point of receiving aperture, under real conditions of propagation of laser radiation in the atmosphere, can be non-Gaussian because of violation of the prerequisites of the central quantum theoreme,[12] for example, because of correlation of the contributions caused by the dependence of the positions of scatterers at single scattering in turbulence,[13] acts of scattering at multiple scattering,[14] phases at propagation of the beam under conditions of turbulence,[15,16] or because of small number of aerosol particles in the scattering volume.[13] In these cases, the results on statistics of photoelectrons and PC obtained for a Gaussian field[17] are incorrect.

Lognormal approximation of the probability density of the radiation intensity[15,16,18,19] has the widest area of application in the cases of non-Gaussian field:

$$f(I) = \frac{1}{\sqrt{2\pi}\sigma I}\exp\left\{-\left[\ln\frac{I}{I_0}+\frac{1}{2}\sigma^2\right]^2 \Big/ 2\sigma^2\right\}, \qquad (4)$$

where $\sigma^2 = \ln(D_I/I_0^2+1)$ is the parameter of the distribution, I_0 is the mean intensity, D_I is its variance. It was noted[15,20] that numerous measurements of the fluctuations of the intensity of light propagated through the turbulent atmosphere, carried out on real atmospheric paths under conditions of both weak fluctuations of the intensity and at their saturation give, in the majority of cases, the distributions close to the logarithmic normal. Some deviations from it are observed in the range of deep freezing and great peaks of the intensity above the mean level. The results of theoretical analysis of fluctuations of the intensity in the range of weak fluctuations[20] also show that its probability density is close to the lognormal distribution.

Some results of investigation of the statistical characteristics of photoelectrons and photocounts for the lognormal distribution of the intensity are presented in Refs. 21 to 25. The probability distribution of the number of photoelectrons $P(n, T)$ is determined[21] under condition that $T \ll \tau_c$ (τ_c is the correlation time of the recorded radiation), which is called the Diament–Teich distribution. The formulas are obtained[22] for the moments of distributions of the number of PE of any order at arbitrary ratio between the time of sampling and the radius of temporal correlation of the field. The results of experimental investigations of the statistical characteristics of photoelectrons are presented in Refs. 23 to 25 for the laser radiation passed through the turbulent medium, and the estimates of the effect of the dead time on the moments of the distribution $P(n, T)$ are obtained in the frameworks of a simple model.

The algorithm for numerical simulation and calculated results for different statistical characteristics of SEP and PC on the set time intervals T at lognormal distribution of the intensity are considered in this paper.

Algorithm for numerical simulating the system of recording optical radiation

Simulation of the recording was carried out using the “intensity – photodetector – SEP flux – PC flux” scheme in the following sequence:

1) simulation of realizations of the power of optical radiation

$$\lambda(t) = \eta\iint\limits_{S^*} I(\mathbf{r},t)\mathrm{d}^2\mathbf{r};$$

2) simulation of the time moments t_i and t_i^* of the occurrence of SEP and PC, respectively;

3) determination of the numbers of SEPs and PCs over the time interval T;

4) processing of the results of the numerical experiment.

The joint probability density of the time moments t_1, t_2, ..., t_n of the occurrence of SEP and PC at the output of the PMT for the specific realization $\lambda(t)$ has the form[5]:

$$f(t_1,t_2,\ldots,t_n|\hat{\lambda}(t)) = \prod_{i=1}^{n}\hat{\lambda}(t_i)\exp\left(-\int_{t_{i-1}}^{t_i}\hat{\lambda}(t)\mathrm{d}t\right), \qquad (5)$$

i.e. t_1, t_2, ..., t_n are independent random values with the conditional probability density

$$f(t_i|\hat{\lambda}(t)) = \hat{\lambda}(t_i)\exp\left(-\int_{t_{i-1}}^{t_i}\hat{\lambda}(t)\mathrm{d}t\right),\ i = 1, 2, ..., n, \quad (6)$$

which is exponential.

Thus, for imitation of the time moments $\{t_i\}$, $i = \overline{1, n}$, it is necessary to preliminary simulate the field of intensity $I(\mathbf{r}, t)$ on the surface of the receiving aperture. Let us consider the asymptotic $S_a \ll s_c$, then $\lambda(t) \approx \eta S_a I(\mathbf{r}_0, t)$, where S_a is the area of the receiving aperture, $\mathbf{r}_0$ is the radius-vector of its center, s_c is the coherence area of the detected radiation. In this case the mean value $\bar{\lambda}$ and the correlation function $B_\lambda(\tau)$ of the random process $\lambda(t)$ are equal to $\eta S_a I_0$ and $(\eta S_a)^2 D_I \rho_I(\tau)$, respectively. Here $D_I = I_0^2(e^{\sigma^2} - 1)$ is the variance of the intensity of the detected radiation, $\rho_I(t)$ is the coefficient of time correlation of intensity. It is easy to generalize the algorithm to the case of $S_a \gg s_c$. To do this, one can use the approach proposed in Refs. 8 and 10. Let us simulate the lognormal random process $\lambda(t)$ with the constant step Δt on the time interval T. As a result, we obtain the set of random values

$$\{\lambda_i = \lambda((i-1)\Delta t)\},\ i = \overline{1, n+1};\ n = T/\Delta t,$$

which can be considered as coordinates of the vector $\eta = \|\eta_1, \eta_2, \ldots, \eta_{n+1}\|^{\mathrm{T}}$ with the lognormal distribution. The algorithm for simulation of such vectors[26] is reduced to imitation of the normally distributed vector $\xi = \|\xi_1, \xi_2, \ldots, \xi_{n+1}\|^{\mathrm{T}}$ with the correlation matrix $\mathbf{B} = \|\rho_{ij}^{(g)}\|$, the elements of which are

$$\rho_{ij}^{(g)} = \ln[1 + (e-1)\rho_{(ij)}]. \quad (7)$$

Here $\rho_{(ij)} = \rho_I[(i-j)\Delta t]$. Simulation of the Gaussian vector ξ is related with the calculation of the eigenvalues and eigenvectors of the correlation matrix $\mathbf{B} = \|\rho_{ij}^{(g)}\|$. To do this, the procedure was used of singular expansion of the matrix.[27] The relationship between η_i and ξ_i $(i = \overline{1, n+1})$ is defined as follows[26]:

$$\eta_i = \exp(\sigma\xi_i + \alpha),$$

where

$$\alpha = \ln(I_0) - \frac{\sigma^2}{2}. \quad (8)$$

To make calculations, it is necessary to specify the form of $\rho_I(\tau)$, approximation of which[20] was considered for different conditions of propagation of laser radiation through turbulent atmosphere. In particular, the possibility has been noted of using, under conditions of strong fluctuations, the Gaussian approximation of the time correlation of intensity $\rho_I(\tau) = \exp(-\tau^2/a^2)$ (a is the parameter) with the radius of time correlation

$$\tau_c = \int_{-\infty}^{+\infty}\rho_I^2(\tau)\mathrm{d}\tau = \sqrt{\frac{\pi}{2}}a. \quad (9)$$

The choice of this approximation is not of principle importance for calculations, and allows us to reveal the dependence of the characteristics of the SEP and PC fluxes on different characteristics of the random process $\lambda(t)$. If necessary, one can also use the proposed algorithm for calculations using other forms of $\rho_I(\tau)$.

A successive simulation of the time moments $\{t_i\}$ with exponential probability density (6) and the time moments $\{t_i^*\}$ of the occurrence of PC is carried out for each $\hat{\lambda}(t)$ realization. Fluctuations of the DT are not taken into account, i.e., overlapping of SEPs with constant width $\Delta\tau$ at the level of the discrimination threshold is considered, as well as their subsequent recording by the paralyzable counter.

The advantage of the numerical experiment is that it enables one not only to study the statistical characteristics of photocurrent at lognormal distribution of the intensity, but also estimating their dependence on the statistics of intensity of the detected radiation. In particular, it is interesting to compare the results calculated for lognormal and exponential probability densities of the intensity in the frameworks of the same experiment. The second case corresponds to the Gaussian field of the detected radiation, for which the statistical characteristics of the SEP and PC fluxes were considered in detail in Refs. 9 to 11.

The structure of the algorithm is not changed at exponential probability density of the intensity

$$f(I) = \frac{1}{I_0}\exp(-I/I_0),$$

corresponding to the Gaussian field statistics. Simulation of the random process $\lambda(t)$ in this case is carried out using the following transform[26]:

$$\lambda(t) = \nu_1^2(t) + \nu_2^2(t), \quad (10)$$

where $\nu_1(t)$ and $\nu_2(t)$ are independent Gaussian random processes with zero mean values and the same correlation functions

$$B_{\nu_1}(\tau) = B_{\nu_2}(\tau) = \frac{\bar{I}S_a\eta}{2}\sqrt{\rho_I(\tau)}. \quad (11)$$

Equality of the variances of the lognormal and Gaussian random processes is reached at the values of the parameter $\sigma = \sqrt{\ln 2} \approx 0.833$.

Calculated results and discussion

Figures 2 to 4 show the results calculated using different statistical characteristics of the SEP and PC fluxes on the time interval $T = 1$ at $S_a = 1$, $\eta = 1$, the PC width at the level of the discrimination threshold $\Delta\tau = 0.1$. In this case, the mean intensity I_0 is numerically equal to the mean value of PCs, $\bar{n}$. Thus set values of the parameters allows one to simplify interpretation of the obtained results with no effect on their generality. In calculations, we set the value of the relative rms deviation of the intensity $\delta_I = \sqrt{D_I}/I_0$, which is related with the parameter of the lognormal distribution σ by the relationship $\sigma^2 = \ln(\delta_I^2 + 1)$. The values σ and the ratio $\mu_3^{\ln}/\mu_3^{\text{ex}}$ of the third central moments $\mu_3^{\ln}$ and μ_3^{ex} of the lognormal and exponential distribution, respectively, are shown in the Table. The value of the relative rms error was checked in calculations, which did not exceed 0.1% in all cases.

For a comparison, dotted lines in Figs. 2 to 4 show the corresponding dependences for the exponential intensity distribution, which corresponds to the Gaussian field of the strength. One should pay attention to different scales of the abscissa axes in Figs. 4*a* and *b*. Therefore, for convenience of comparison of the plots, the probability distribution $P(m)$ calculated for $\delta_I = 1$ is also shown in Fig. 4*a*.

Let us consider in a more detail interpretation of the obtained results. First of all, it is the effect of the distribution of the intensity fluctuations on the statistical characteristics of the number of SEP and PC. Let us compare the dependences $\bar{m}$ and $D(m)$ in Figs. 2*a* and *b* at lognormal ($\delta_I = 1$) and exponential intensity distributions, which have the same mean values and variances. The value of the relative change in $\bar{m}$ at passing from one intensity distribution to the other at $\bar{n} = 5$ is 0.06, and it is equal to 0.11 at $\bar{n} = 10$. The relative change in $D(m)$ is 0.13 and 0.14, respectively. These differences are caused by the fact that the third central moments μ_3 of the considered intensity distributions are different. It is seen in the Table that the value of the ratio $\mu_3^{\ln}/\mu_3^{\text{ex}} = 2$ at $\delta I = 1$, and it does not depend on the mean intensity I_0.

The discussed distinctions between the dependences of the mean value and the variance of the numbers of PCs show the contribution of statistical characteristics of the intensity fluctuations to the corresponding characteristics of the fluxes of photocounts. The variances $D(n)$ in Fig. 2*b* at different statistics of the intensity coincide, in the considered case, with each other due to the equality of the time correlation coefficients $\rho_I(\tau)$ and variances D_I. The difference between the dependences revealed in calculations lied within the limits of the error of the numerical experiment. This fact was used when testing the algorithm of numerical simulation.

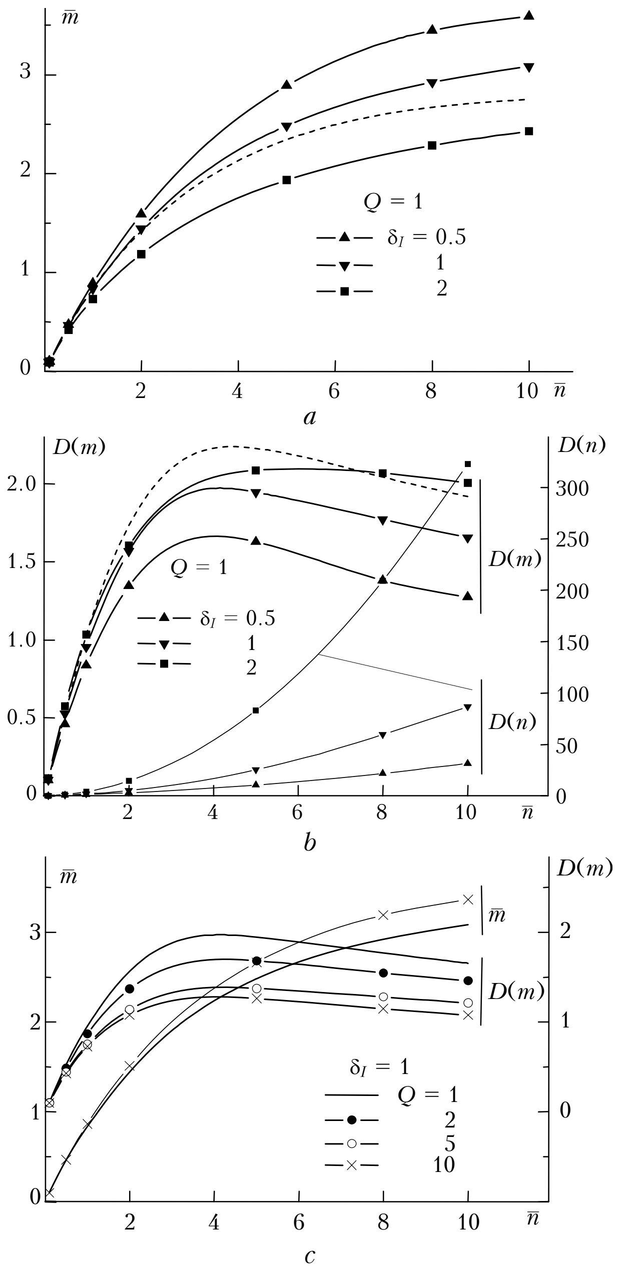

Fig. 2. Dependences of the mean value $\bar{m}$ (*a*, *c*) and variances $D(n)$ and $D(m)$ (*b*, *c*) of the SEP and PC numbers on $\bar{n}$ at different Q and δ_I. Dotted lines show the dependences for exponential distribution of the intensity, solid lines for the lognormal one.

Table. The values of the parameter σ of the lognormal distribution and the ratio $\mu_3^{\ln}/\mu_3^{\text{ex}}$ as functions of the values of the relative rms deviation, δ_I, of the intensity

$\delta_I = \sqrt{D_I}/I_0$	0.5	0.8	1	1.2	1.5	2
σ	0.472	0.703	0.833	0.944	1.086	1.269
$\mu_3^{\ln}/\mu_3^{\text{ex}}$	0.102	0.745	2	4.603	13.289	55.983

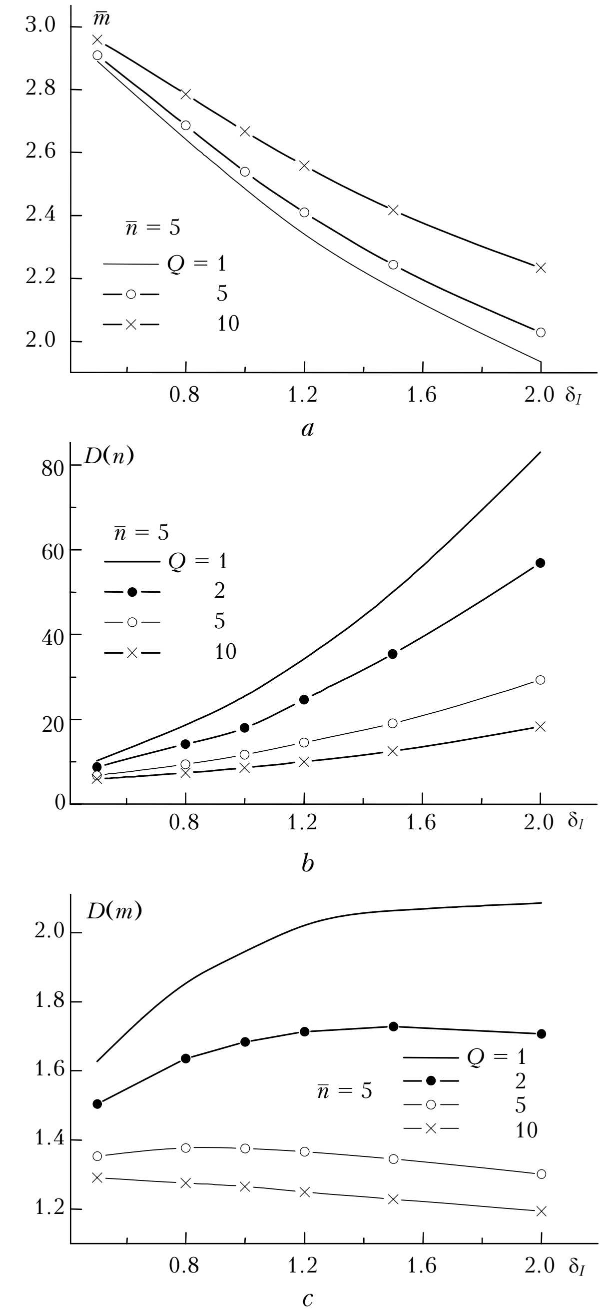

Fig. 3. Dependences of the mean value $\bar{m}$ (*a*) and the variances $D(n)$ and $D(m)$ of the numbers of SEP and PC on δ_I.

The probability distributions of the SEP and PC numbers for lognormal at $\delta_I = 1$ and exponential (dotted line) intensity distributions are shown in Fig. 4 for a comparison. In spite of the equality of the variances of SEP numbers in the considered case, the probability distributions $P(n)$ differ from each other (see Fig. 4*a*), that shows their dependence on the nature of the intensity fluctuations.

Comparison of the variances of the SEP and PC numbers in Figs. 2*b* and 3*b* and *c* enables one to draw a conclusion that the effect of "dead" time leads to the decrease of fluctuations of the SEP number, i.e., affects as a smoothing filter. Hence, $D(n) > D(m)$, and this difference increases with the increasing the intensity of the radiation recorded.

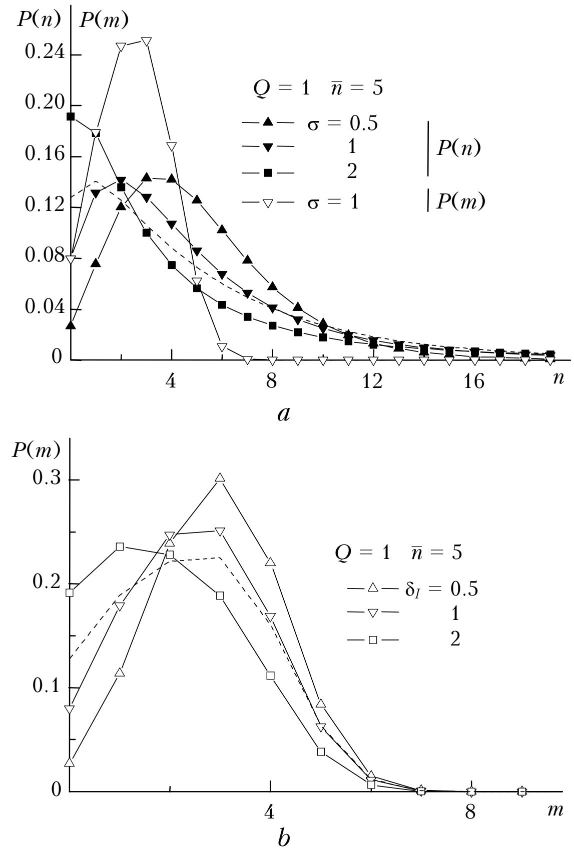

Fig. 4. Probability distributions of the SEP $P(n)$ and PC $P(m)$ numbers at different δ_I. Dotted lines show the dependences for exponential distribution of intensity, solid lines for the lognormal distribution.

The result shown in Fig. 3*a* is interesting. It is seen that the increase of the intensity variance related to the growth of the parameter δ_I leads to a decrease of the mean PC number at a given $\bar{n}$ value. Explanation of such an unexpected, at the first sight, dependence is related with the fact that the ratio $\mu_3^{\ln}/\mu_3^{\rm ex}$ increases with the increase of δ_I (see Table), i.e., the third central moment of the exponential distribution of intensity $\mu_3^{\ln}$, increases, for which the condition $\mu_3^{\ln} > 0$ is fulfilled. As a result, the depth of "positive fluctuations" of intensity increases, that leads to a stronger grouping of SEP and stronger, due to this fact, effect of the "dead" time.

As to the dependence of $\bar{m}$ and $D(m)$ on the parameter Q, one should note that its increase is related to the decrease of the correlation time, i.e., the "period" of the intensity fluctuations. Then, the time intervals of SEP grouping and, hence, the effect of the "dead" time decrease. This leads to an increase of $\bar{m}$ (see Fig. 3*a*) and decrease of the variances $D(n)$ and $D(m)$ (see Figs. 3*b* and *c*). The Poisson asymptote $D(n) \to \bar{n}$ takes place at $Q \to \infty$. The dependence of $D(m)$ on δ_I at different values of the parameter Q is quite complicated, that is illustrated in Fig. 2*c*.

The statistical characteristics of the SEP and PC fluxes at lognormal intensity distribution have been studied numerically. It was shown that the nature of the intensity fluctuations makes significant contribution to the statistical characteristics of the SEP and PC fluxes. One can use the obtained data for interpretation of the results on weak radiation fluxes measured using the photon counting.

Acknowledgments

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