Representation of a solution of the admixture transport equation and its applications

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We discuss a representation of solution of atmospheric pollution transport equation in terms of a reduction of initial equation to the classical one-dimensional heat transfer equation. The altitude behavior of the horizontal wind velocity components and turbulent diffusion coefficient is assumed to be known. The solution of the boundary value problems exhibits two-functional arbitrariness. The arbitrary functions are determined from known pollutant distribution at two levels, along a preset line; then, the parametric representation of spatial concentration field is found. The efficiency of this approach rests upon the fact that, due to introduction of new variables, the flow region in the plane is canonic, as well as the fact that numerous available (both analytical and numerical) results can be used to solve one-dimensional Fourier equation. In the initial variables, the concentration distribution is determined through a simple recalculation using simple conversion formulas of transition from Cartesian coordinates to new ones.

Introduction

Presently, among different ecological problems, the most important is the problem on prediction of the spatial distribution of industrial aerosol. A promising approach to studying aerosol transport and transformation is the method based on application of turbulent diffusion equations[1] whose solution is sought for the near-ground atmospheric layer. This raises the problem of obtaining either continuous or discrete pollutant (aerosol) concentration distribution satisfying some boundary conditions.

This paper is aimed at seeking an approximate analytical solution of a stationary three-dimensional (3D) parabolic-type equation which models the advective-diffusion transport of a scalar substance in the shear flow.

Transformation of pollutant transport equation

Let us consider the following equation[1]:

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=\frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right), \qquad (1)$$

where u and v are projections of horizontal wind velocity; and k_z is turbulent diffusion coefficient, for all of which it is assumed that

$$u=U(x,y)\tanh^2(\alpha z),\ \ v=V(x,y)\tanh^2(\alpha z);$$

$$k_z=k_0\left[U^2(x,y)+V^2(x,y)\right]\tanh^2(\alpha z). \qquad (2)$$

Here $U(x, y)$ and $V(x, y)$ are arbitrary conjugate harmonic functions; and k_0 and α are constants;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad\Rightarrow\quad \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0,$$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial U}{\partial y}-\frac{\partial V}{\partial x}=0.$$

In equation (1), in place of variables x and y we introduce new variables φ and ψ, defined by the differential relations

$$d\varphi=U\,dx+V\,dy,\ \ d\psi=-V\,dx+U\,dy. \qquad (3)$$

In new variables, Eq. (1) will read

$$\frac{\partial c}{\partial\varphi}=k_0\left(\frac{\partial^2 c}{\partial z^2}+\frac{2\alpha}{\sinh(\alpha z)\cosh(\alpha z)}\frac{\partial c}{\partial z}\right). \qquad (4)$$

Let us point out one property of this equation. Let $c_0(\varphi,\psi,z)$ be a solution of this equation; then

$$c(\varphi,\psi,z)=f_1(\psi)\,c_0(\varphi+f_2(\psi),\psi,z)$$

will also be a solution of the equation (4) with arbitrary functions $f_1(\psi)$ and $f_2(\psi)$. Owing to this fact we can, using known solutions of equation (4) as a starting point, construct a series of exact solutions for the equation (4) and, correspondingly, for the initial equation (1).

According to Refs. 3 and 4, solution of the equation (4) is expressed via the solution of the classic heat transfer equation

$$\frac{\partial L}{\partial\varphi}=k_0\frac{\partial^2 L}{\partial z^2} \qquad (5)$$

in the form of the first-order differential equation

$$c = L - \frac{1}{\alpha}\coth(\alpha z)\frac{\partial L}{\partial z}. \qquad (6)$$

Equations (4)–(6) show that the 3D problem for the initial equation (1) is reduced to the 1D one. From expression (5) it follows that, if $L_0(\varphi, \psi, z)$ is the solution of this equation, then

$$L = L_0\left[\varphi + f_1(\psi), \psi, z + f_2(\psi)\right],$$

where $f_1(\psi)$ and $f_2(\psi)$ are arbitrary functions, will also be a solution of this equation. Thus, we can point out to the two-functional arbitrariness in solution of equation (4), irreducible to that found earlier.

To close the 1D problem, we need to specify the distribution of concentration along the lines

$$\varphi = \varphi_0,\ \ \psi = 0\ \ c(\varphi_0, 0, z) = c_0(z),$$

$$\varphi = \varphi_0,\ \ z = 0\ \ c(\varphi_0, \psi, 0) = c_0(\psi). \qquad (7)$$

When the solution with two-functional arbitrariness is chosen, we also need to know the distribution

$$c(\varphi_0, \psi, H) = c_H(\psi),$$

where H is the height of the studied region.

To pass from the boundary-value problem for equation with variable coefficients (4) and (7) to the corresponding boundary-value problem with constant coefficients (5), the known boundary values $c(\varphi, \psi, z)$ are to be used to determine the corresponding values $L(\varphi, \psi, z)$. For this, equality (6) is presented as

$$\frac{\partial}{\partial z}\frac{L}{\cosh(\alpha z)} = -\frac{\alpha\sinh(\alpha z)}{\cosh^2(\alpha z)}c. \qquad (8)$$

Integration of this relation yields

$$L = -\cosh(\alpha z)\int_a^z \frac{\alpha\sinh(\alpha\tau)}{\cosh^2(\alpha\tau)}c\,d\tau + A(\varphi,\psi)\cosh(\alpha z), \quad (9)$$

where $A(\varphi, \psi)$ is an arbitrary integration function. Because of the linearity of equation (5), the second summand in Eq. (9) must satisfy it.

This gives

$$\frac{\partial A}{\partial \varphi} = k_0\alpha^2 A,\ \ A = B(\psi)e^{k_0\alpha^2\varphi}$$

with an arbitrary function $B(\psi)$. Because the concentration $c(\varphi, \psi, z)$ and $L(\varphi, \psi, z)$ are limited, this dictates that $A(\varphi, \psi) = 0$; $a \to \infty$, and, thus,

$$L(\varphi,\psi,z) = \cosh(\alpha z)\int_z^\infty \frac{\alpha\sinh(\alpha\tau)}{\cosh^2(\alpha\tau)}c(\varphi,\psi,\tau)\,d\tau. \quad (10)$$

Since $c(\varphi_0, 0, z)$ is known, the corresponding boundary value that follows from Eq. (10) is

$$L(\varphi_0,0,z) = \cosh(\alpha z)\int_z^\infty \frac{\alpha\sinh(\alpha\tau)}{\cosh^2(\alpha\tau)}c(\varphi_0,0,\tau)\,d\tau. \quad (11)$$

To determine the boundary value $L(\varphi_0, \psi, z)$, let us remove the indefiniteness in representation (6). We have

$$c(\varphi,\psi,0) = L(\varphi,\psi,0) - \frac{1}{\alpha^2}\frac{\partial^2 L}{\partial z^2}(\varphi,\psi,0),$$

and hence

$$L(\varphi,0,0) = Ae^{\alpha^2 k_0\varphi} + \alpha^2 k_0 e^{\alpha^2 k_0\varphi}\int_{-\infty}^{\varphi} e^{-\alpha^2 k_0\tau}c(\tau,0,0)\,d\tau,$$

where A is an arbitrary constant. The finiteness of $L(\varphi, 0, 0)$ at $\varphi \to \infty$ results in $A = 0$.

If a distribution of concentration $c_H(\varphi, 0, H)$ is known, we arrive at the boundary condition of the third kind

$$\frac{\partial L}{\partial z}(\varphi,0,H) - \alpha\tanh(\alpha H)L(\varphi,0,H) =$$

$$= \alpha\tanh(\alpha H)c(\varphi,0,H) = \qquad (12)$$

The boundary conditions for $L(\varphi, \psi, z)$ are thus determined and so we are led to the mixed boundary value problem for Fourier equation (5), where $L(\varphi, 0, 0) = L_0(\varphi)$ is assumed as an the initial condition.

The transform used above and known as Boussinesq transform is of interest by itself. Its applications are not exhausted by equations of the type (1). It also can be used for studying the complete system of equations of heat–mass transfer since it transforms nonorthogonal grid to orthogonal one. This, in turn, simplifies the construction of finite-difference equations.

An example of analytical solution of pollutant transport problem

We will use these results to determine the distribution of pollution generated by Gaussian-type source at bypassing of a circular hill modeled as a circular cylinder $(x - m)^2 + y^2 = R^2$. The complex potential of the flow is known and reads[2]:

$$w = \varphi + i\psi - \varphi_0 = \frac{v_\infty}{2}\left[(Z - m) + \frac{R^2}{Z - m}\right],$$

$$Z = x + iy, \qquad (13)$$

where φ_0, v_∞, m, and R are known constants.

For a Gaussian jet model[1]:

$$L = \frac{M}{\sqrt{2\pi\varphi}}\left[\exp\left(-\frac{(\xi+h)^2}{4\alpha^2 k_0\varphi}\right) + \exp\left(-\frac{(\xi-h)^2}{4\alpha^2 k_0\varphi}\right)\right],$$

$$h = \alpha H,\ \xi = \alpha z. \qquad (14)$$

Then, for concentration c we have the presentation

$$c = \frac{MA(\psi)}{\sqrt{2\pi\varphi}}\left\{\left[\exp\left(-\frac{(\xi+h)^2}{4\alpha^2 k_0\varphi}\right) + \exp\left(-\frac{(\xi-h)^2}{4\alpha^2 k_0\varphi}\right)\right] + \right.$$

$$+\coth\xi\left[\frac{\xi+h}{2\alpha^2 k_0\varphi}\exp\left(-\frac{(\xi+h)^2}{4\alpha^2 k_0\varphi}\right) + \right.$$

$$\left.\left. + \frac{\xi-h}{2\alpha^2 k_0\varphi}\exp\left(-\frac{(\xi-h)^2}{4\alpha^2 k_0\varphi}\right)\right]\right\}. \qquad (15)$$

The arbitrary function $A(\psi)$ is determined from the condition $c_0(\psi) = c(\varphi_0, \psi, 0)$, with the distribution of concentration along the line $\varphi = \varphi_0$, $z = 0$ assumed to be known. Removing the indefiniteness in the second term of Eq. (15), we find

$$c_0(\psi) = \frac{MA(\psi)}{\sqrt{2\pi\varphi_0}}\left[1 + \frac{1}{2\alpha^2 k_0\varphi_0}\left(1 + \frac{h^2}{2\alpha^2 k_0\varphi_0}\right)\right]\times$$

$$\times\exp\left(-\frac{h^2}{4\alpha^2 k_0\varphi_0}\right). \qquad (16)$$

The complex potential of the flow will be considered to conform to the law (13), corresponding to bypass of the circle of radius R by a translational flow with the speed v_∞ at infinity.

Then, relations (13), (15), and (16) give us the sought spatial distribution of pollution from a single point-like source in a parametric form, though in this case the dependence of concentration on Cartesian coordinates can be expressed explicitly.

The distribution of pollutant concentration during bypass of a cylinder by dust-polluted flow (with the distance between pollution source and center of cylinder $m = 100$ m; radius of cylinder $R = 50$ m; $v_\infty = 5$ m/s; $H = 50$ m; $k_0 = 0.1$ m^2/s; and $\alpha = 1.2$) is shown in Fig. 1.

Figure 1 presents dependences of maximum concentration on coordinate z, as well as curves showing the dependence of pollutant concentration on x- and z-coordinates. The results presented indicate that the variations of concentration depend significantly on z-level. Near the earth surface at some distance from the source, the concentration is maximum. The greater z, the closer the maximum to the source. At the level of the pollutant ejection $z = H$ the concentration monotonically decreases with growing x. At altitudes higher than the altitude of ejection there again appears a maximum of c at some distance x.

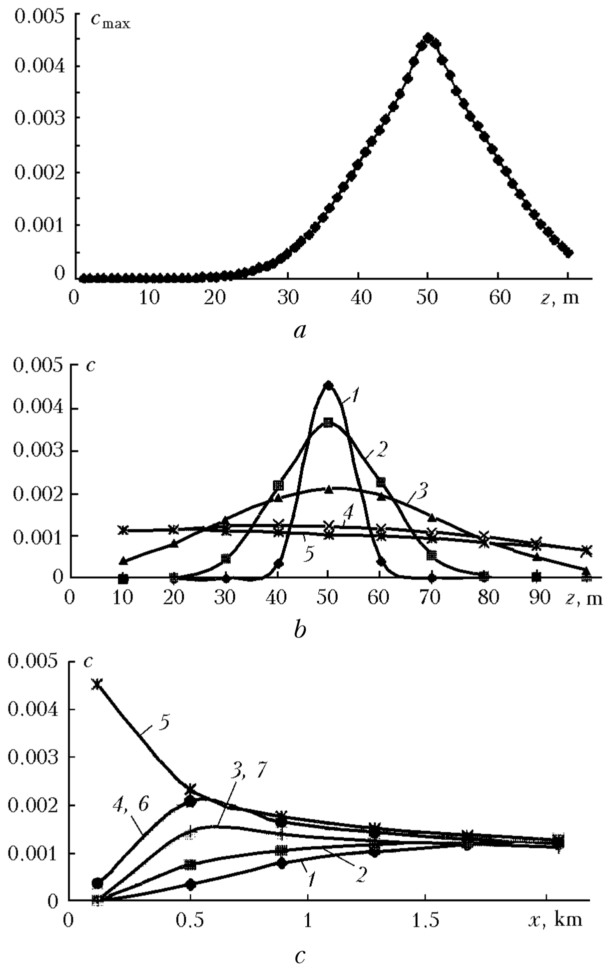


Fig. 1. Dependence of maximum concentration c_{max} on z (a); c on x: $x = 0.12$ (curve *1*); 0.2 (curve *2*); 0.6 (curve *3*); 2 (curve *4*); 4 km (curve *5*) (b), and c on z: $z = 10$ (curve *1*); 20 (curve *2*); 30 (curve *3*); 40 (curve *4*); 50 (curve *5*); 60 (curve *6*); and 70 m (curve *7*) (c).

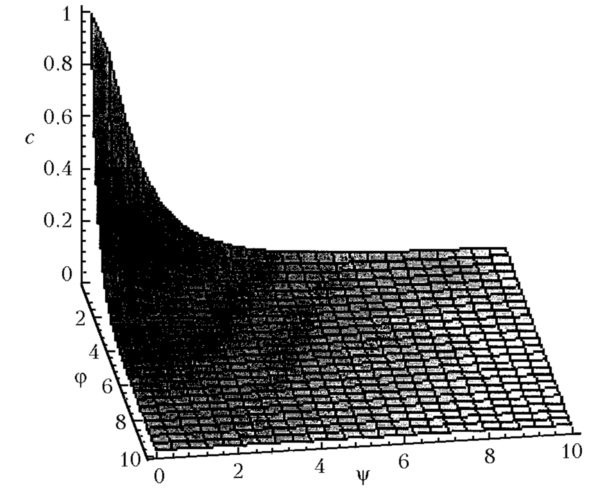
The calculated data can also be used to estimate the vertical profile of concentration as a function of distance from the source. When the distance x is short, the altitudinal maximum is observed at approximately source level $z = H$. As x increases, the concentration maximum descents, so that distribution of concentration in altitude looses symmetry: the concentration level near the surface is higher than that in the upper air layers.

Pollutant distribution in a street canyon

We also considered the task of the spread of pollutant from a source, located in free space, into a channel of width $2a$. In this case, the variables x, y and φ, ψ are related as

$$x + iy = \frac{\varphi + i\psi}{u} + \frac{a}{\pi}\exp\left(-\frac{\pi(\varphi + i\psi)}{au}\right), \qquad (17)$$

where u is the velocity at infinity.



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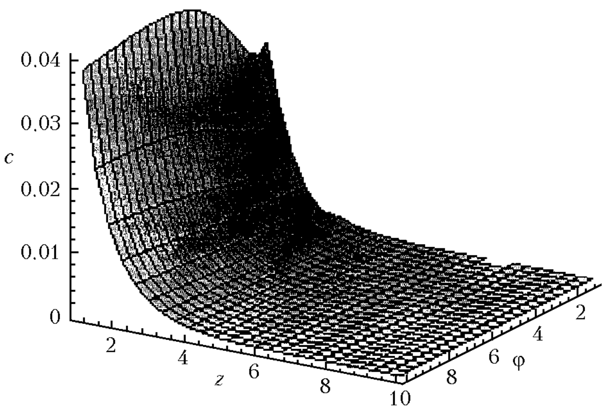

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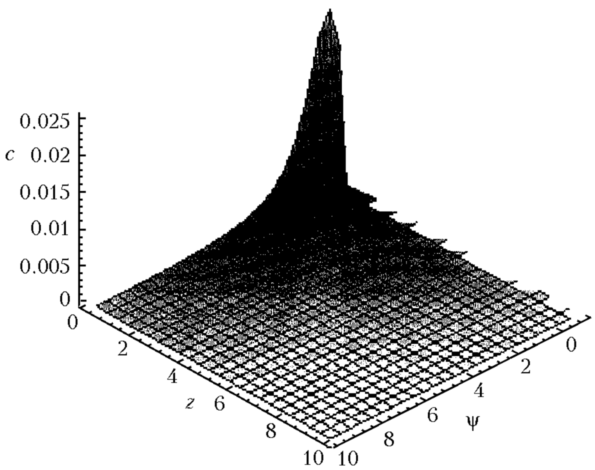

c

Fig. 2. Concentration values for $z = 0$ (*a*), $\psi = 0$ (*b*), and $\varphi = 0.1$ (*c*) with $\alpha = 0.1$, $k_0 = 0.1$ m^2/s, and $l = 1$.

If the initial distribution $L(\varphi, \psi, z)$ is specified in the form

$$L(\varphi,\psi,0) = \frac{\exp(-l\varphi)}{\psi^2+1},\; L(0,\psi,z) = \frac{\exp(-mz)}{\psi^2+1}, \quad (18)$$

then for $L(\varphi, \psi, z)$ we obtain the expression

$$L(\varphi,\psi,z) = \frac{1}{\psi^2+1}\int\limits_{\frac{z}{\sqrt{k_0\varphi}}}^{\infty} \exp\left\{-\left[l\left(\varphi - \frac{z^2}{k_0 s^2}\right) + \frac{s^2}{4}\right]\right\} \mathrm{d}s +$$

$$+\frac{1}{\psi^2+1}\int\limits_0^{\infty}\left[\exp\left(-\frac{(z-\xi)^2}{4k_0\varphi} - m\xi\right) - \right.$$

$$\left. -\exp\left(-\frac{(z+\xi)^2}{4k_0\varphi} - m\xi\right)\right]\mathrm{d}\xi. \quad (19)$$

Then, the relations (6), (17), and (19) will yield the spatial distribution of concentration, so that

$$c(\varphi,\psi,0) = \left(1 + \frac{l^2}{\alpha^2 k_0}\right)\frac{\exp(-l\varphi)}{\psi^2+1}.$$

The results of calculations using the formulas mentioned above are presented in Fig. 2.

Conclusion

We presented one of the methods of reducing dimensionality of the initial problems, affording an efficient way of analytical and numerical solution of equation considered here.

In addition, the transformation used converts nonorthogonal (in plane) grids to orthogonal ones, substantially simplifying the arrangement of numerical approaches for complete pollutant transport equations.

References

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