ADAPTIVE METHODS FOR IMAGE RESTORATION (A REVIEW)

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We present a review of adaptive methods for solving the problem of "vision" through a turbulent atmosphere. The problem is formulated, the model of the light field is proposed, and two approaches to the adaptation based on the measurements of phase distortions and the usage of the sharpness functions are described. The well-known sharpness functions are considered, the analysis of these functions with respect to absolute maximum is performed. The Bayes approach to adaptive image restoration is described. It is shown that only some of varieties of sharpness functions have absolute maxima which are relevant for image restoration. The ways for finding the absolute maximum are examined. The possibility of using sharpness functions are studied in successive seeking the absolute maximum over segments of the adaptive element. It is shown that such a technique is inapplicable for sharpness functions with secondary maxima. Some conclusions are drawn on the use of one or another sharpness functions under different conditions.

1. INTRODUCTION. THE PROBLEM OF "VISION" THROUGH THE ATMOSPHERE

When the objects are observed in astronomy and laser detection and ranging, we are frequently faced with the fact that the turbulent atmosphere, through which the observations are performed, distorted the information about the object shape which is embedded in the front of the wave reflected from or emitted by these objects. When the telescopic image is formed, a turbulent atmosphere causes limitations of the angular resolution of an optical system varying from 1 to 5 seconds of arc for the visible spectral range.[1] The problem of vision through the atmosphere has been a subject of interest for science and technology developed rapidly in recent years. Not only the methods of classical optics but also the highly developed methods of quantum electronics and statistical radio–engineering were employed to solve the aforementioned problem. As a result, new methods of processing the light fields have arisen, which differ from the traditional telescopic reception.

The methods of vision through a turbulent atmosphere are conventionally classified as interferometric, holographic, and adaptive methods.[2] The interferometric methods are used for determining the geometric parameters of the objects which emit or scatter incoherent radiation. The majority of these methods rely on the determination of square modulus of the spatial frequency spectrum of the object[3–5] or the object itself[6–9] based on which, following the Van-Zittert-Zernike theorem, an image of the object or its autocorrelation function is formed. These methods have a significant disadvantage: the data processing is a very intricate process, which would entail lengthy computations. In speckle interferometry it takes 0.1–1 s to record a hundred of independent speckle images. Coherent radiation is used in holographic methods that also makes it possible to restore the autocorrelation function[10] and the image[11] of the objects. However, the most of holographic methods require (which is rarely realistic) a reference point source which is located in the zone isoplanar with the object under study in order to compensate for the atmospheric phase distortions. The adaptive methods of vision through a turbulent atmosphere are more promising than the interferometric and holographic methods since they enable us to form the diffraction limited images both in natural and coherent light with laser beam illumination. The images in this case are formed in real time which is smaller than the time during which the atmosphere can be assumed frozen. The main idea of adaptation is as follows. As is well known,[12] the effect of the atmosphere on the radiation emitted from an object located in the isoplanar zone can be described by the amplitude-phase screen approximation. If a phase transparency, which compensates for the phase fluctuations caused by the atmosphere, is placed in front of the receiving aperture, then the resolution of the optical system will be nearly diffraction limited due to the fact that amplitude fluctuations of the signal from the object weakly distort the image.[13] The correction using a controllable phase transparency should be accomplished during the time in which the atmosphere can be assumed frozen since the state of the atmosphere is a function of time. This method is called the adaptive optics method.

To compensate for phase distortions of the desired signal caused by a turbulent atmosphere, it is necessary to find them. This is the most critical point of adaptation. The most widely employed are the two approaches, one is based on the usage of the wavefront sensors and the other — on the sharpness function maximization.[14]

In the first approach phase distortions of the optical field are measured in the plane of the entrance or exit pupil of the optical receiving system. As is well known in optics, according to the specific properties of square–law detection, the phase cannot be measured directly, therefore the distributions of the phase difference are usually' recorded. The methods of measuring the phase difference can be both direct and indirect. The direct methods employ the segmentation of the entire area of the pupil and the formation of the unresolved image of the object on each of these segments. The undistorted image energy centers of gravity of the point object (a plane wave is incident on the receiving optical system) form a coordinate grid, given that the local tilts are observed for any elementary segment, the image energy center of gravity shifts from the corresponding

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node of the grid proportionally to this tilt. The recalculation of the vector shifts of the image from the nodes of the grid makes it possible to determine a local tilt for any segment which is proportional to the phase difference between the edges of the segment in discrete representation. This is a so-called Hartmann sensor.[25] By joining the phase differences, one may determine the wavefront phase.[14]

The indirect methods are based on the interference of the optical field and either the reference wave or the optical field itself shifted at a vector in the pupil plane. Recording of interferograms with the use of a reference wave is seriously impeded by the limited temporal coherence of an optical field of the object and cannot virtually be used in the wavefront sensors. The most generally applicable are the modified Michelson interferometers[10] and the shearing interferometers[22,24] which allow one to obtain the interference pattern of the optical field self-action. The first case is used to obtain the information about the object itself and the seconc case is intended for measuring the distributions of the wavefront phase differences. The distributions sought are calculated from the curvature of the interference bands of any elementary section of the pattern. Two interferograms with the shift of the field in two mutually orthogonal directions make it possible to determine the corresponding phase differences. As in the previous case, we determine the wavefror.t phase by joining these differences. After the required processing, the measured phase is applied to the operating mechanisms (e.g., pushers) of the adaptive device (mirror or phase transpare*ncy*1. As a result, it matches the wavefront form.

The second approach relies on varying wavefront phase due to control with the segments of the adaptive element during the transmission of the optical field. Various functionals, called the sharpness functions, are formed based on the measured intensity of the transmitted field. Tho variation in the wavefront phase caused by the adaptive element results in corresponding variation of the value of the functional. Maximum or minimum of the sharpness function must be reached by means of compensation for phase distortions by an adaptive element, that is, for a "sharp" image. Since there are no direct measurements of phase distortions in this approach, the most difficult problem is the choice of the sharpness function and the methods of control by the adaptive element.

To discuss the adaptive methods in further-details, the light field model is examined first.

2. LIGHT FIELD MODEL. WAVEFRONT SEHSORS

The coherent and incoherent fields should be examined independently. If the object under study is illuminated by monochromatic radiation with the wavelength λ, then the light field scattered by the object in the receiving plane ρ has the form[15]

$$\varepsilon_S(\rho,t) = \mathrm{Re}\,\varepsilon_0(\rho)\exp(-i\omega t), \quad (2.1)$$

where $\omega = 2\pi c/\lambda$ is the circular frequency, c is the velocity of light, t is time, and $\varepsilon_0(\rho)$ is the complex function which describes a spatial structure of the field.

With such distances R from the object to the ,-e.je; ving plane that the condition $R^3 \gg \frac{\pi}{4\lambda}\,|\rho* - r|^4$ is satisfied for all $\rho \notin \Omega$ and $r \notin \Omega$, the function $\varepsilon(\rho)$ is related to the complex field amplitude $E(r)$ in the image plane of the object r as follows[16]:

$$\varepsilon(\rho) = \int_\Omega E(r)\,G(r-\rho)\exp\{\Phi(r,\rho)\}\,dr, \quad (2.2)$$

whore $G(r-\rho) = (i\lambda R)^{-1}\exp\{i\kappa R + \frac{i\kappa}{2R}\,||r-\rho|^2\}$, Ω is the projection of the object onto the image plane, the function $\Phi(r,\rho)$ describes the signal distortions caused by the propagation of radiation from the point with the coordinate r through the etnio-pbere to the point with the coordinate ρ. The function $\Phi(r,\rho) = \chi(r,\rho) + i\varphi(r,\rho)$ is complex: $\chi(r,\rho)$ is the logarithm of amplitude distortions and $\varphi(r,\rho)$ are the phase distortions of the signal. Since a great number of random and independent factors affect the radiation propagating through a turbulent atmosphere, the functions $\chi(r,\rho)$ and $\varphi(r,\rho)$ obey the Gaussian statistics by virtue of the central limit theorem. If the observed object is located in the zone of a.n isoplcjiar atmosphere, then to describe the effect of the atmosphere, we can make use of the &.mpl itude-phase screen approximation $\Phi(r,\rho) = \Phi(\rho)$, given that the screen is placed in the receiving plane. In some cases (e.g., on the vertical paths) amplitude fluctuations occur to be weak $\chi(\rho) \approx 0$), and one can consider the phase screen approximation $\Phi(r,\rho) = i\varphi(\rho)$, that is

$$\varepsilon(\rho) = \exp\{i\varphi(\rho)\}\cdot\int_\Omega E(r)\,G(r-\rho)\,dr. \quad (2.3)$$

In the phase screen approximation for telescopic reception in the image plane x, the intensity is recorded

$$I(x) = \frac{1}{2}|E(x)|^2 = \frac{1}{2}\left|\int_\Omega E(r)g(r,x)dr\right|^2, \quad (2.4)$$

where

$$g(r,x) = S_0(\lambda^2 Rz)^{-1}\exp\left\{ik(z+R) + \frac{ik}{2z}|x|^2 + \frac{ik}{2R}|r|^2\right\}g_1^*(r+\frac{R}{z}x), \quad (2.5)$$

$$g_1(r+\frac{R}{z}x) = S_0^{-1}\int_{-\infty}^{\infty}W(\rho)\exp\left\{i\phi(\rho) + \frac{ik}{R}(r+\frac{R}{z}x)\rho\right\}d\rho, \quad (2.6)$$

$f^{-1} = z^{-1} + R^{-1}$, f is the focal length of the telescope, z is the distance from the aperture plane to the image plane, S_0 is the aperture area, and $w(\rho)$ is the aperture function which is equal to unity inside the aperture and zero outside it. If there are no phase distortions in the signal ($\varphi(\rho) = 0$) and the number of resolution elements of the object image obtained with the receiving optics $M_0 = \frac{S\,S_0}{(\lambda R)^2} \gg 1$ (S is the area of the object projection onto the image plane Ω), then the width of the function $g_0(r) = g_1(r)\,|_{\varphi(\rho)=0}$ is much smaller than the linear dimension of the zone Ω. In this case for the object with a specular surface, when the field on the object J(r) is described by a smooth slowly varying function, as compared with the function $g_0(r)$, we observe the image

$$I(x) = \frac{R^2}{2z^2}\left|E(-\frac{R}{z}x)\right|^2. \quad (2.7)$$

If the surface of the object is rough, the complex amplitude of the field in the image plane of the object is a realization of the randomly δ–correlated Gaussian process with zero mean, that is,

$$<E(r)> = 0,$$

$$<E(r_1)E(r_2)> = <E^*(r_1)E^*(r_2)> = 0,$$

and

$$<E(r_1)E^*(r_2)> = <E^*(r_1)E(r_2)> = u(r_1)\delta(r_1 - r_2), \quad (2.8)$$

where the function $u(r)$ is proportional to the field intensity in the image plane and <...> refers to averaging over an ensemble of realizations of the surface microstructure of the object. The average distribution of the image intensity obtained for the illumination of a rough object by monochromatic radiation agrees with distribution of tlje image intensity in incoherent light[2]

$$<I(x)> = \frac{1}{2} S_0^2 (\lambda Rz)^{-2} \int\limits_{\Omega} u(r)\left|g_1(r+\frac{R}{z}x)\right|^2 dr = I_0(x) * h(x), \quad (2.9)$$

that is, represents the convolution of the undistorted image $I_0(x)$ with pulse response of the system "receiving aperture – atmosphere" $h(x) = |g_1(x)|^2$. When there are no phase distortions, the average image intensity is equal to

$$<I(x)> = \frac{1}{2} S_0 (\lambda^2 z^2)^{-1} u(-\frac{R}{z}x).$$

Every individual realization of the image represents the average image modulated by a randomly speckle pattern with a speckle contrast which is equal to unity. In some cases, a speckle structure of images should be smoothed out. To do this, one may use the illumination of the rough object by radiation with different wavelengths, a method of spatial averaging in the image plane, and scanning radiation.[18] The resulting intensity of the smoothed image is described by relations (2.9) and (2.10).

In practice, the receiving light is polychromatic, that is, it has a finite spectral width $\Delta\lambda$. The spectral width is determined by the width of the spectral line of the illumination (laser radiation or natural light) and by a spectral filter used in the image plane. Since the phase distortions caused by the atmosphere are different for different wavelengths, a sufficiently narrow band $\Delta\lambda$, within the limits of which the images are efficiently compensated for distortions, is required for adaptation. A change of $\Delta\lambda$ in the wavelength leads to a change of the order of $\delta_\varphi \sim \Delta\lambda/\lambda$ in the phase distortions at the receiving aperture, where the phase standard deviation $\delta_\varphi = 10$–25 rad (Ref. 19). This provides for the condition $\Delta\lambda/\lambda \le 0.03$–$0.01$. In the case of such a polychromatic signal and an object with a macrosurface lying in the image plane and a microsurface, whose roughness is smaller than the length of radiation coherence $\lambda^2/\Delta\lambda \sim 30$–$100\,\lambda$, when the condition $\Delta\lambda M_0^{1/2}/\lambda \ll 1$ holds valid, the formula (2.4) remains valid, and a speckle structure of the image preserves. If a diffuse object is three-dimensional and the deviation of its surface from the image plane within the region of the element of optical resolution is greater than the length of radiation coherence, then the speckle structure of the image is smoothed out. The width of the radiation spectrum $\Delta\nu = c\Delta\lambda/\lambda^2$ corresponds to the coherence length $(\Delta\nu)^{-1} c$. For observation of three-dimensional objects, the speckle contrast of the image is $(c/d\Delta\nu)^{1/2}$, where d is the size of a characteristic deviation of the object surface from the image plane within the region of the element of optical resolution.

Let us now proceed to the treatment of the wavefront sensors. As was indicated in the Introduction, the approach of the wavefront sensors relies on the step-by-step measurement of phase differences over the receiving aperture area. The information about the measured phase differences is processed and used to form the controlling signals for an adaptive element. ' A shearing interferometer[22,2]* and a Hartmann sensor are most generally applicable. In the plane of an input aperture of the shearing Interferometer the interferogram of the fields *c(p)* and e(p + J), where *1* is the vector of the shift, is recorded. For coherent illumination from this interferogram we determine the value

$$\mathrm{Re}\,\varepsilon(\rho)\,\varepsilon^*(\rho + 1) = |\varepsilon_0(\rho)|\,|\varepsilon_0(\rho + 1)|\,\cos[\varphi(\rho) -$$

$$- \varphi(\rho + 1) + \arg\varepsilon_0(\rho) - \arg\varepsilon_0(\rho + 1),$$

where $\varepsilon_0(\rho) = \varepsilon(\rho)|_{\varphi(\rho)=0}$ is the object field in the receiving plane. It is not difficult to notice that the phase distortions caused by the atmosphere can be found when we are aware of the phase of the object field $\arg\varepsilon_0(\rho)$, i.e., when the function $E(r)$ is known. For incoherent Illumination, the value

$$\mathrm{Re}\,\varepsilon(\rho)\,\varepsilon^*(\rho + 1) =$$

$$= \mathrm{Re}\int\limits_{\Omega} u(r)G(r-\rho)G^*(r-\rho+1)dr \sim$$

$$\sim \cos\left[\varphi(\rho) - \varphi(\rho+1) - |l|^2 - 2\rho l + \arg\int\limits_{\Omega} u(r)\exp(i\frac{k}{R}rl)\,dr\right].$$

is found.

In so doing, if $U(r)$ is the arbitrary central–symmetric function, then $\arg\int\limits_{\Omega} u(r)\exp(i\frac{k}{R}rl)\,dr = 0$. In general, determining the phase difference $\varphi(\rho) - \varphi(\rho + 1)$, would call for a knowledge of the function $U(r)$. In the Hartmann sensor the received radiation is incident on a matrix of small (as compared with the correlation length of phase distortions) lenses, each of which forms a poorly resolved image. The local wavefront tilt at each of the lenses is found from the position of the image energy center of gravity. For a lens with the center at the point $\rho = 0$, phase distortions gradient is estimated by the formula

$$\left(\mathrm{grad}\,\hat{\varphi}(\rho)\Big|_{\rho=0}\right) = \frac{k}{z}\,\frac{\int_{-\infty}^{\infty} xI(x)dx}{\int_{-\infty}^{\infty} I(x)dx}.$$

Using the relations (2.4) and (2.9) for the image intensity, where $\varphi(\rho) = \rho\,(\mathrm{grad}\,\varphi(\rho)|_{\rho=0})$, it is not difficult to show that in the case of δ–shaped or even functions $E(r)$ and $u(r)$ for coherent and incoherent illumination the intensity distribution is symmetric relative to the point $x = \frac{k}{z}\,\mathrm{grad}\,\varphi(\rho)|_{\rho=0}$. Then the estimate of the wavefront tilt will be well-founded. In the remaining cases, this estimate causes an error. To determine and eliminate the error, the knowledge of the functions $E(r)$ and $U(r)$ is required.

Thus, in the general case, when the wavefront sensors are used to measure atmospheric phase distortions, a reference object of the known shape with the prescribed functions $E(r)$ or $u(r)$, e.g., a reference point is required. Otherwise, the distortions caused by the atmosphere cannot be determined.

3. ADAPTIVE COMPENSATION FOR DISTORTIONS WITH THE USE OF SHARPNESS FUNCTION OF THE IMAGE

As mentioned in the introduction, the other approach to the image adaptation consists of forming an image by maximizing the "sharpness function".[26–29] The wavefront sensor is not used in this method. The correction of the wavefront is implemented by continuous control of the individual segments of an active optical element on the basis of maximizing the value called a sharpness function. Some functionals of the measured intensity (in this case, the image intensity) are called sharpness functions. Maximization of sharpness functions is believed to provide compensation for atmospheric phase distortions. This approach, as well as the choice of concrete sharpness functions, is of heuristic character and is not universal for the problem of image restoration. Let us assume that the receiving aperture provides for high optical resolution $M_0 \gg 1$.

When there is no a *priori* information on the shape of an extended object, the functional[26]

$$S_1 = \int\limits_{\Omega_p} I^2(x)dx, \tag{3.1}$$

is often used. Here, Ω_p is the zone of recording of the intensity in the image plane, S_p is the area of this zone. Following Eq. (2.4), the image intensity is

$$I(x) = \frac{1}{2}(\lambda^2 Rz)^{-2} \int\int\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2) \times$$
$$\times \exp\left\{i\psi(\rho_1) - i\psi(\rho_2) - i\frac{k}{z}x(\rho_1 - \rho_2)\right\}d\rho_1 d\rho_2,$$

where

$$\varepsilon_1(\rho) = \int\limits_{\Omega} E(r)\exp\left\{\frac{ik}{2R}|r|^2 - i\frac{k}{R}r\rho\right\}dr = i\lambda R\exp\left\{-\frac{ik}{2R}|\rho|^2\right\}\varepsilon_0(\rho)$$

is a function of the object field $\varepsilon_0(\rho)$ and $\psi(\rho) = \varphi(\rho) + \theta(\rho)$ and $\theta(\rho)$ are phase distortions in the aperture plane caused by the adaptive element. Taking account of the fact that with a fairly large zone of recording $S_p \gg (\lambda z)^2/S$ (this condition follows from the relations $M_0 \gg 1$ ans $S_p \sim (\frac{z}{R})^2 S_0$, the expression

$$\exp\left\{-i\frac{k}{z}x\rho\right\}dx \sim (\lambda z)^2\delta(\rho)$$

is valid, for the sharpness function S_1 for coherent illumination we find

$$S_1 = \frac{1}{4}(\lambda^3 R^2 z)^{-2}\iiint\limits_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3) \times$$
$$\times \varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2)\varepsilon_1(\rho_3)\varepsilon_1^*(\rho_1 - \rho_2 + \rho_3)\exp\Big\{i\psi(\rho_1) -$$
$$- i\psi(\rho_2) + i\psi(\rho_3) - i\psi(\rho_1 - \rho_2 + \rho_3)\Big\}d\rho_1 d\rho_2 d\rho_3 \le$$
$$\le \frac{1}{4}(\lambda^3 R^2 z)^{-2}\iiint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3) \times$$
$$\times |\varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2)\varepsilon_1(\rho_3)\varepsilon_1^*(\rho_1 - \rho_2 + \rho_3)|d\rho_1 d\rho_2 d\rho_3. \tag{3.2}$$

The sharpness function S_1 is maximum, if for any ρ_1, ρ_2, and ρ_3 the expression

$$\psi(\rho_1) - \psi(\rho_2) + \psi(\rho_3) - \psi(\rho_1 - \rho_2 + \rho_3) +$$
$$\arg\Big\{\varepsilon_1(\rho_1)\varepsilon_1(\rho_2)\varepsilon_1(\rho_3)\varepsilon_1^*(\rho_1 - \rho_2 + \rho_3)\Big\} = \text{const}$$

is valid. Taking $\rho_1 = \rho_3 = \rho$ and $\rho_2 = \rho + x$ and approaching x to zero, we find that the second derivative $[\varphi(\rho) + \arg\varepsilon_1(\rho)]'' = 0$ or

$$\psi(\rho) + \arg\varepsilon_1(\rho) = a + b\rho, \tag{3.3}$$

where a and b are arbitrary constants. This does not provide for compensation for phase distortions, the phase distortions caused by the atmosphere re cannot be separated from the phase of the object field. For incoherent illumination, Eq. (3.2) is replaced by the following equation:

$$S_1 = \frac{1}{4}(\lambda^3 R^2 z)^{-2}\iiint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3) \times$$
$$\langle\varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2)\rangle\langle\varepsilon_1(\rho_3)\varepsilon_1^*(\rho_1 - \rho_2 + \rho_3)\rangle\exp\Big\{i\psi(\rho_1) -$$
$$- i\psi(\rho_2) + i\psi(\rho_3) - i\psi(\rho_1 - \rho_2 + \rho_3)\Big\}d\rho_1 d\rho_2 d\rho_3 =$$
$$= \frac{1}{4}(\lambda^3 R^2 z)^{-2}\iiint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3) \times$$
$$\times \left|\int\limits_{\Omega_p} dr u(r)\exp\left\{i\frac{k}{R}r(\rho_2 - \rho_1)\right\}\right|^2 \times \exp\Big\{i\psi(\rho_1) -$$
$$- i\psi(\rho_2) + i\psi(\rho_3) - i\psi(\rho_1 - \rho_2 + \rho_3)\Big\}d\rho_1 d\rho_2 d\rho_3 \le$$
$$\le \frac{1}{4}(\lambda^3 R^2 z)^{-2}\iiint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3) \times$$
$$\times \left|\int\limits_{\Omega} u(r)\exp\left\{i\frac{k}{R}r(\rho_2 - \rho_1)\right\}dr\right|^2 d\rho_1 d\rho_2 d\rho_3,$$

that gives the condition of maximum

$$\psi(\rho) = a + b\rho. \tag{3.5}$$

Hence, for incoherent illumination, the maximization of the sharpness function S_1 allows us to restore the undistorted images of the extended objects with an arbitrary shape.

The functional

$$S_2 = \int_{\Omega_p} \left[\frac{\partial^{n+m} I(x_1, x_2)}{\partial x_1^n \partial x_2^m} \right] dx_1 dx_2, \quad (3.6)$$

where x_1 and x_2 are the coordinates of the vector x and n and m are the integers,[26] is similar to the considered sharpness function S_1. Differentiation with respect to the coordinates x_1 and x_2 leads to an additional real integrand multiplier $\left(\frac{k}{z}\right)^{2n+2m} (\Delta\rho)_1^{2n} \cdot (\Delta\rho)_2^{2m}$ in Eqs. (3.2) and (3.4), where $(\Delta\rho)_1$ and $(\Delta\rho)_2$ are the coordinates of the vector $\Delta\rho = \rho_2 - \rho_1$. The arguments do not change in the process. There also exists the sharpness function[26]

$$S_3 = \int_{\Omega_p} I^n(x) dx, \quad (3.7)$$

where n is the integer and $n > 2$. Let us consider the case when $n = 3$. By analogy with the functional S_1, it is easy to derive the condition of .maximum for coherent illumination

$$\psi(\rho_1) - \psi(\rho_2) + \psi(\rho_3) - \psi(\rho_4) + \psi(\rho_5) - \psi(\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5) + \arg\{\varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2) \times \varepsilon_1(\rho_3)\varepsilon_1^*(\rho_4)\varepsilon_1(\rho_5)\varepsilon_1^*(\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5)\} = \text{const}$$

for any ρ_1, ρ_2, ρ_3, ρ_4, and ρ_5. Taking $\rho_1 = \rho_3 = \rho_4 = \rho_5 = \rho$ and $\rho_2 = \rho + x$, we again obtain relation (3.3). For incoherent illumination, the condition of maximum

$$\psi(\rho_1) - \psi(\rho_2) + \psi(\rho_3) - \psi(\rho_4) + \psi(\rho_5) - \psi(\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5) + \arg\{\langle\varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2)\rangle \times \langle\varepsilon_1(\rho_3)\varepsilon_1^*(\rho_4)\rangle\langle\varepsilon_1(\rho_5) \times \varepsilon_1^*(\rho_1 - \rho_2 + \rho_3 - \rho_4 + \rho_5)\rangle\} = \text{const}$$

leads to relation (3.5) for only the central–symmetric objects, when $u(r) = u(-r)$. In the general case, the argument of the product in the last equality is nonzero, and maximization of the functional S_3 does not satisfy the condition (3.5). The sharpness functions for n = 4, 5,... are treated in the same way as for n = 3. As a result, we can conclude that maximization of S_3 is applicable for point and central–symmetric objects for incoherent illumination.

The sharpness function[26] with an amplitude mask $M(x) = I_0(x)$ is written in the form

$$S_4 = \int_{\Omega_p} M(x) I(x) dx,$$

where $I_0(x)$ is the intensity of the undistorted image. For the incoherent illumination and for an arbitrary mask $M(x)$ the functional will be

$$S_4 = \frac{1}{2}(\lambda^2 Rz)^{-2} \iint_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\exp\{i\psi(\rho_1) - i\psi(\rho_2)\} \times \int_{\Omega_p} M(x)\exp\{-i\frac{k}{z}x(\rho_1 - \rho_2)\}dx \times \int_{\Omega_p} u(r)\exp\{-i\frac{k}{R}r(\rho_1 - \rho_2)\}dr d\rho_1 d\rho_2. \quad (3.9)$$

Since the mask correspond to the undistorted image $M(x) = I_0(x) \sim u[-\frac{R}{z}r)$, the integrals over the variable x and r in Eq. (3.9) are complex conjugate, and their product is real. Therefore, S_4 is maximized for $\psi(\rho)$ = const. If the mask is not relevant for the object shape, but the functions $M(x)$ and $u(r)$ are central symmetric functions, then we can also compensate for phase distortions. For example, the sharpness functions[26]

$$S_5 = \int_{\Omega_p} |x|^2 I(x) dx, \quad (3.10)$$

and

$$S_6 = I(x_0) \quad (3.11)$$

satisfy the above indicated requirement. Here x_0 is the point lying in the image plane. For coherent illumination of the diffuse object, when the mask $M(_x) = \langle I_0(x)\rangle$ is used, the condition of maximization of S_4 is obtained by functional differentiation of Eq. (3.8):

$$\psi(\rho) + \arg\varepsilon_1(\rho) + \arg\left\{\int_{-\infty}^{\infty}\int_{\Omega} \omega(\rho_1)\varepsilon_1^*(\rho_1) \times \exp\{-i\psi(\rho_1) - i\frac{k}{R}(\rho_1 - \rho_2)r\}u(r)dr d\rho_1\right\} = 0. \quad (3.12)$$

In the general case, the function $\psi(\rho)$, being the solution of Eq. (3.12), does not result in the compensation for phase distortions. The similar result for $M(x) = |E(-\frac{R}{z}x)|^2$ is obtained for coherent illumination of the specular object.

$$\psi(\rho) + \arg\varepsilon_1(\rho) + \arg\left\{\iiint_{-\infty}^{\infty}\int_{\Omega} \omega(\rho_1)\varepsilon_1^*(\rho_1) \times \exp\{-i\psi(\rho_1) - i\frac{k}{R}(r_1\rho_1 - r_2\rho_2)\} \times E(r_1)E^*(r_2)dr_1 dr_2 d\rho_1\right\} = 0.$$

The functional

$$S_7 = \int_{\Omega_p} |I(x) - I_0(x)|^2 dx, \quad (3.13)$$

which characterize the rms error, is expressed in terms of the sharpness functions S_1 and S_4;[26] therefore, all what has been said above is valid too. To use the sharpness function S_4, we need for an *a priori* information about the object shape. When the *a priori* information is insufficient, the authors of Ref. 29 suggest that the iteration procedure be employed, that is, the estimator mask $M_k(x) = I_{k-1}(x)$ is to be formed on the basis of the image intensity $I_{k-1}(x)$ measured in the $(k - 1)$th step and consequently used for maximizing the functional

$$S_5 = \int\limits_{\Omega_p} |x|^2 I(x)\mathrm{d}x, \qquad (3.14)$$

in the kth step. The rate and efficiency of the convergence of the iteration process are assumed to be dependent on the amount of an *a priori* information about the object (estimates of the coordinates and the shape of the object). The more complete the information about the object, the better are the results obtained using this algorithm and conversely. However, the convergence of iteration procedure was not investigated in Ref. 29.

The well–known sharpness functions are not successfully employed because the radiation scattered by the object is unsuitable for measuring the atmospheric phase fluctuations, since it is impossible to distinguish between the phase distribution of the object field and the phase fluctuations. The supplemental information (similar to $M(x)$) is required for their separation. The polarization characteristics of the light fields reflected from the object are used in Ref. 30 as a supplemental information for adaptive restoration of images. When convex objects with rough surfaces are illuminated by linearly polarized coherent radiation, the reflection patterns for parallel and cross polarization differ.[31] If a point of the convex object has the coordinate $r = 0$, then the maximum of the parallel component of reflected intensity $u_{xx}(r)$ and minimum of the cross component $u_{yy}(r)$ are observed at this joint. The function $Q(r) = u_{xx}(r) / Au_{yy}(r)$ for the objects of such shapes (viz. sphere, ellipsoid or cone) has a typical δ–function peak at $r = 0$. Phase distortions are compensated for by maximizing the sharpness function

$$S_9 = \frac{I_x(0)}{AI_y(0)}, \qquad (3.15)$$

where $I_x(x)$ and $I_y(x)$ are the intensities of images obtained for different polarizations and A is the ratio of the integral intensities I_x and I_y. When there are no phase distortions, the sharpness function $S_9 = Q(0)$ reaches its maximum since it has the δ–function peak shape. When atmospheric distortions occur, this peak is smoothed out and its value and "sharpness" define a degree of adaptive compensations for distortions. Unfortunately, the algorithm exhibits significant disadvantages: it is suitable for only limited class of convex objects, the recording of a cross component I_y dictates, due to its smallness, the use of a high–sensitivity sensor. Moreover, the performance with S_9 is possible only for smoothed (averaged) intensities I_x and I_y, since their speckle–structures are different.[12]

4. ADAPTATION BASED ON SHARPNESS FUNCTIONS OF THE SPATIAL FREQUENCY SPECTRUM

The masks in the Fourier plane (spatial frequency spectrum) can be used in addition to the masks in the image plane. For instance, by analogy with S_4, we can propose

$$S_{10} = \left| \int\limits_{\Omega_\Phi} M(f)F(f)\mathrm{d}f \right|^2, \qquad (4.1)$$

where Ω_Φ is the zone of recording of the image Fourier spectrum, f is the spatial frequency, $M(f)$ and $F_0(f($, where $F(f) = F_0(f)$, are distorted and undistorted spatial spectra of images, respectively. It is easy to show that maximization of S_{10} results in the compensation for phase distortions only for incoherent illumination of the objects. With the lack of an *a priori* information about the spectrum of the object $F_0(f)$, as in the case of S_8, we can make use of an iteration algorithm with the sharpness function

$$(S_{11}) = \left| \int\limits_{\Omega_\Phi} M_k(f)F_k(f)\mathrm{d}f \right|^2, \qquad (4.2)$$

where the estimator mask $M_k(f) = F_{k-j}(f)$ is used in the kth step.

The sharpness functions S_{10} and S_{11} do not exploit those advantages which provide for the transition to the Fourier plane. It should be noted that for incoherent illumination, by virtue of Eq. (2.9), the Fourier spectrum will have the form

$$F(f) = \int\limits_{\Omega_p} I(x)e^{2\pi i f x}\mathrm{d}x = F_0(f)H(f)$$

and

$$F_0(f) = \int\limits_{\Omega_p} I_0(x)e^{2\pi i f x}\mathrm{d}x, \qquad (4.3)$$

where

$$H(f) = S^{-1}\int\limits_{-\infty}^{\infty} w(\rho)\omega(\rho - z\lambda f) \times$$
$$\times \exp\left[i\psi(\rho) - i\psi(\rho - z\lambda f)\right]\mathrm{d}\rho \qquad (4.4)$$

is the atmosphere–lens optical transfer function (OTF). As can be seen from Eq. (4.3), the distortions which enter . H(Я are associated with the spectrum not by the integral, as for the image, but by the product with $F_0(f)$. The sharpness function

$$S_{12} = \frac{1}{|F_0(0)|^2}|F_0(f_0)H(f_0)|^2 \qquad (4.5)$$

was proposed in Ref. 33. Here f_0 is spatial frequency and $0 < |f_0| < \frac{\rho_0}{z\lambda}$ (ρ_0 is the correlation length of the phase fluctuations of the field $\varepsilon(\rho)$ caused by the turbulent atmosphere). It is possible to show that the absolute maximum of the function (4.5) is reached when $\psi(\rho) - \psi(\rho - z\lambda f_0) = \text{const}$, that is, the condition (3.5) holds. Actually, the atmosphere–lens OTF at a frequency of f_0 is maximum in the absence of phase perturbations.[34]

In the coherent case, by virtue of Eq. (2.4), the Fourier spectrum of the image represents the autocorrelation function of the field:

$$F(f) = \int_{\Omega_p} I(x)e^{2\pi i f x}dx = S^{-1}\int_{-\infty}^{\infty} \omega(\rho)\omega(\rho - z\lambda f)\varepsilon_1(\rho) \times$$

$$\times \varepsilon_1^*(\rho - z\lambda f)\exp\left[i\psi(\rho) - i\psi(\rho - z\lambda f)\right]d\rho. \qquad (4.6)$$

As a result, the sharpness function S_{12} is maximized, when

$$\arg \varepsilon_1(\rho) - \arg \varepsilon_1(\rho - z\lambda f_0) + \psi(\rho) - \psi(\rho - z\lambda f_0) = \text{const}$$

that is, the condition (3.3) holds and the image can not be restored.

To restore the images in the coherent case,the authors of Ref. 35 propose the object be successively illuminated by two coherent waves during the time in which the atmosphere can be considered "frozen". Due to the interference of these waves, the image, in the absence of distortions, would be modulated by spatial harmonic with a frequency of f_0 determined by the geometry of preliminary illumination and by a single wave. To perform adaptive compensation for distortions, the sharpness function

$$S_{13} = \left| \int_{\Omega_p} \frac{I_2(x) + n_0}{I(x) + n_0} \exp(2\pi i f_0 x)dx \right|^2 \Bigg/ \left| \int_{\Omega_p} \frac{I_2(x) + n_0}{I(x) + n_0} dx \right|^2, \qquad (4.7)$$

is used, where $I_2(x)$ is the intensity of the distorted image for illumination by two waves

$$I_2(x) = \left| \int_{\Omega} E(r)\left[\exp(i\varphi_1 + i\pi f_0 r) + \exp(i\varphi_2 + i\pi f_0 r)\right] g\left(\frac{r}{R} + \frac{x}{z}\right) \right|^2 \qquad (4.8)$$

and n_0 is the additive background preliminary illumination. As shown in Ref. 35, at small n_0 the absolute maximum S_{13} is reached when $g(\frac{r}{R} + \frac{x}{z}) = g_0(\frac{r}{R} + \frac{x}{z})$ in the direction $(\frac{r}{R} + \frac{x}{z}) \parallel f_0$.

If the object is illuminated by two pairs of coherent waves, which produce two interference patterns in the mutually orthogonal directions, then the phase fluctuations are also compensated In these directions, and the condition $\psi(\rho)$ = const holds, this allows us to restore the image.

5. THE BAYES ADAPTIVE APPROACH TO MEASUREMENTS OF AND COMPENSATION FOR ATMOSPHERIC PHASE DISTORTIONS

As is well known, a light signal is usually random due to random signal distortions caused by the atmospheric inhomogeneities, the quantum nature of radiation recording, the intrinsic receiver noises, the random external background, and many other factors. Therefore, strictly speaking, the measurement of and compensation for atmospheric phase distortions represent a statistical problem. To solve this problem, we must turn to the statistical theory. The appropriate method was proposed by Wald[36] and was called the decision theory. It relies on an optimal solution called the Bayes solution and obtained from minimization of the mean risk criterion. This approach is extremely fruitful in developing optimal algorithms for processing the signals[37] and is successfully employed to synthesize the algorithms of optimal processing of the light fields[38] distorted by the atmosphere.

The basic principle of the adaptive Bayes approach discussed in detail in the Ref. 39 is as follows. We form the estimate of the mean *a posteriori* risk criterion and minimize it under an *a priori* uncertainty conditions based on the results of observations with the help of an appropriate choice of the decision rule. The authors of Refs. 29 and 40–42 employed such an approach to the problem of compensation for atmospheric distortions $\varphi(\rho)$. The adaptive Bayes decision rule for the case of parametric dependence of the *a posteriori* risk on the collection of the parameters φ is reduced to the determination of the estimates of maximum likelihood $\vec{\varphi}$ as a function of the sample from apparent realization δ. The sample ε is described by the sequence of values $\varepsilon_1, \ldots, \varepsilon_k$ observed at time $t_1,\ldots,t_k$, that is, $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_k\}$. If the likelihood functional, which incorporates the information obtained by the time t_k, is denoted by $P_k(\varepsilon_1, \ldots, \varepsilon_k|\varphi$ then the estimate of the maximum likelihood $\varphi(\varepsilon_1, \ldots, \varepsilon_k)$ is found from the equation

$$P_k(\varepsilon_1, \ldots, \varepsilon_k|\hat{\varphi}) = \max_{\varphi} P_k(\varepsilon_1, \ldots, \varepsilon_k|\varphi). \qquad (5.1)$$

The similar equation for the natural logarithm of the likelihood functional $L_k(\varphi) = \ln P_k(\varepsilon_1, \ldots, \varepsilon_k|\varphi)$ is of the form

$$L_k(\hat{\varphi}) = \max_{\varphi} L_k(\varphi). \qquad (5.2)$$

The estimates of maximum likelihood are found from a system

$$\nabla_\varphi L_k(\varphi) = 0. \qquad (5.3)$$

provided that the partial derivatives exist. Here ∇_φ is the operator of gradient with respect to the components of the vector $\vec{\varphi}$. In the majority of cases of interest to us the solution of Eq. (5.3) has not been found because of its mathematical complexity. For this reason the approximate recursion methods are often used.[29,39–42] In this case the logarithm of the likelihood functional is represented as

$$L_k(\varphi) = L_{k-1}(\varphi) + l_k(\varphi), \qquad (5.4)$$

where $L_{k-1}(\varphi) = \ln P_{k-1}(\varepsilon_1, \ldots, \varepsilon_k|\varphi)$ is the natural logarithm of the likelihood functional for a set of observational data without the last quantity and $l_k(\vec{\varphi}) = \ln P_k(\varepsilon_k|\varepsilon_1, \ldots, \varepsilon_{k-1}, \vec{\varphi})$ is the natural logarithm of conditional probability density ε_k for the given values $\varepsilon_1, \ldots, \varepsilon_{k-1}, \varphi$. Then from Eq. (3.5) we derive

$$\nabla_\varphi L_{k-1}(\varphi) + \nabla_\varphi l_k(\varphi) = 0. \qquad (5.5)$$

If the left side of Eq. (5.5) is expanded into the Taylor series around the value of φ corresponding to the estimate $\hat{\varphi}_{k-1}$ in the $(k-1)$th step and only the linear terras are preserved, then the solution is

$$\hat{\varphi}_k = \hat{\varphi}_{k-1} + D_k^{-1} z_k, \quad (5.6)$$

where

$$z_k = \nabla_\varphi\, l_k(\hat{\varphi}_{k-1}) = \{\partial l_k(\hat{\varphi}_{k-1})/\partial\varphi_1, \ldots, \partial l_k(\hat{\varphi}_{k-1})/\partial l_N\}. \quad (5.7)$$

$$D_k = -\|\partial^2 L_{k-1}(\hat{\varphi}_{k-1})/\partial\varphi_1\partial\hat{\varphi}_j\| - \|\partial^2 l_k(\hat{\varphi}_{k-1})/\partial\varphi_1\partial\varphi_j\| \quad (5.8)$$

is the symmetrical matrix, and $\varphi_1 (I = 1, \ldots, N)$ are the components of the vector φ.

By means of the above approach the optimal estimates of atmospheric phase distortions were found in Refs. 29 and 42 when the point and extended spatio–coherent and spatio–incoherent objects were observed. The case of coherent radiation was studied. For the phase distortions $\varphi(\rho)$ in the plane of the receiving aperture the following model was employed:

$$\varphi(\rho) = \sum_{n=1}^{N} \varphi_n \tilde{\omega}_n(\rho - \rho_n^0),$$

$$\tilde{\omega}_n(\rho - \rho_n^0) = \begin{cases} 1, & \rho \in \Delta_n, \\ 0, & \rho \notin \Delta_n, \end{cases} \quad (5.9)$$

where $\varphi_n = \varphi(\rho_n^0)$ could take any values in the segments Δ_n $(n = 1, \ldots, N)$, which the receiving aperture was divided into. For observation of the point object with the coordinate $\vec{r}_0$, the optimal estimate of phase distortions to within $2\pi n_1$ (n_1 is the integer) is equal to

$$\varphi_n = \arg\varepsilon_n, \quad (5.10)$$

where

$$\varepsilon_n = \frac{A_p}{2N_0}\int_{-\infty}^{\infty}\tilde{\omega}_n(\rho - \rho_n^0)G^*(r_0 - \rho)\varepsilon(\rho)d\rho,$$

$$\varepsilon(\rho) = \int_0^T \varepsilon(\rho, t)e^{i\omega t}dt,$$

$\varepsilon(\rho, t)$ is the field at the receiving aperture, $\varepsilon(\rho, t) = \varepsilon_n(\rho, t) + n(\rho, t)$, $n(\rho, t)$ is the adaptive noise being a random process δ–correlated in space and time with zero mean, N_0 is the noise power, T is the observation time, A_p is the amplitude of the field of the point source, and $\varepsilon_n(\rho, t)$ and $G(r - \rho)$ are determined from Eqs. (2.1) and (2.3). For an object with a specular surface (a spatio–coherent scattered field) an optimal estimate is also found from Eq. (5.10), where

$$\varepsilon_n = \frac{A_0}{2N_0}\int_{-\infty}^{\infty}\tilde{\omega}_n(\rho - \rho_n^0)\varepsilon_0^*(r)\varepsilon(\rho)\,d\rho,$$

$$\varepsilon_0(\vec{\rho}) = \int_\Omega E(r)G(r - \rho)dr,$$

and A_0 is the amplitude of the radiation incident on the specular object. It follows from what has been said above that the optimal estimate of phase distortions is equal to the phase difference between the observed field and the object field averaged over the observation time and the segment Δ_n, that is, in order to measure the object, field distortions it is necessary to perform matched filtering of the object field. It has therefore become necessary to know the shape of the object $E(r)$ in order to estimate the distortions $\varphi(\rho)$, that is, in the absence of an *a priori* information about the object being observed the estimate (5.10) is of low efficiency. Based on the results obtained, the authors of Ref. 29 proposed the sharpness function for the adaptive processing of the field

$$S_{14} = \left|\int_{-\infty}^{\infty}\omega(\rho)\varepsilon_1(\rho)\varepsilon_1^*(\rho)\exp\left(i\psi(\rho)\right)d\rho\right|^2, \quad (5.11)$$

which means the transmission of the object field through a matched filter with the transmission $T(\rho) \sim \varepsilon_1^*(\rho)$, when the field phase $\varepsilon_1(\rho)\exp(i\psi(\rho))$ is matched. The maximization of S_{14} is apparently achievable, when $\arg\varepsilon_1(\rho) + \psi(\rho) - \arg\varepsilon_1(\rho) = \text{const}$, that is, when $\psi(\rho) = \text{const}$. It is also evident that the required knowledge of the shape of this object $E(r)$ makes the usage of the sharpness function S_{14} of low interest. It was suggested in Ref. 29 that under condition of the *a priori* uncertainty the iteration algorithm, similar to Eq. (3.14) for the image plane, be employed: from the field $\varepsilon_{1k-1}(\rho)$ measured in the $(k-1)$th step the estimate for the matched filter $T_k(\rho) \sim \varepsilon_{k-1}^*(\rho)$ is formed in such a way that the sharpness function be maximized in the kth step

$$(S_{15})_k = \left|\int_{-\infty}^{\infty}\omega(\rho)\varepsilon_{1k}(\rho)\varepsilon_{1k-1}^*(\rho)\exp\left(i\psi(\rho)\right)d\rho\right|^2. \quad (5.12)$$

It is assumed that the closer the estimate $T_k(\rho)$ to the actual distribution $\varepsilon_{k-1}^*(\rho)$, the more efficient is the algorithm. However, the convergence of Eq. (5.12) was not studied.

The sharpness function S_{16} of the type (5.11) was proposed in Ref. 43, in which a reference wave with a plane wavefront $\varepsilon^{0*}(\vec{\rho}) = b_0\exp(-i\frac{\kappa}{R}r_0\rho)$ rather than $\varepsilon_1^*(\rho)$, where b_0 is the wave amplitude, was used

$$S_{16} = b_0^2\left|\int_{-\infty}^{\infty}\omega(\rho)\varepsilon_1(\rho)\exp\left(-i\frac{k}{R}r_0\rho + i\varphi(\rho)\right)d\rho\right|^2. \quad (5.13)$$

The maximization of S_{16} apparently makes it impossible to compensate for atmospheric distortions, but leads to the condition (3.3), where $b = -\frac{\kappa}{R} r_0$ and $a = 0$. The problem of the rough surface object (a spatio–incoherent field) has not been solved in an explicit form, and the subsequent recursion formula

$$\hat{\varphi}_{n,k} = \hat{\varphi}_{n,k-1} - \frac{4(\lambda R)^2 N}{kT_0^2 S_0 \int\limits_{\Omega} V(r)u(r)dr} \times$$

$$\times \int\limits_{\Omega} V(\vec{r})\mathrm{Re}\left[i\sum_{m\neq n}^{N} \exp\left(i\hat{\varphi}_{m,k-1}\right)\varepsilon^{*}_{mk}(r) \times\right.$$

$$\left.\times \exp\left(-i\hat{\varphi}_{n,k-1}\right)\varepsilon_{nk}(r)\right]dr, \qquad (5.14)$$

was found from Eqs. (5.5)–5.8). Here, the subscript κ is responsible for the observation time $(\kappa - 1)\,T_0 - kT_0$, $T_0 = T/N_1$, N_1 is the number of observation time intervals, the index n corresponds to the segment Δ_n,

$$V(r) = \frac{u(r)}{2N_0^2\left[1 + T_0 u(r)/4N_0\right]},$$

$$\varepsilon_{nk}(r) = \int\limits_{-\infty}^{\infty} \hat{\omega}_n(\rho - \rho_n^0)\varepsilon_{0k}(\rho)G^{*}(r-\rho)d\rho,$$

and

$$\varepsilon_{0k}(\rho) = \int\limits_{(k-1)T_0}^{kT_0} \varepsilon(\rho, t)e^{1\omega t}dt.$$

This formula was derived based on the assumption that one of the conditions $M_0 \gg N$ or $M_0 \ll N$ is satisfied, where M_0 is the number of elements of optical resolution of the object in the absence of the atmospheric phase distortions, that is, when the phase distortions and the wave from the object have different scales of characteristic variation. The final result for the most interesting situation $M_0 \sim N$ has not been obtained. The recursion formula (5.14) is not universal for several reasons. First, the object shape (function $u(r)$) is to be known. Second, the problems of convergence of the solution (5.14) and of the choice of a zero–order estimate for phase distortions $\hat{\varphi}_{n0}$ were not studied in Refs. 29 and 40–42, although in some cases the use of this formula is inefficient. For instance, in the case of low level noise for a zero–order estimate $\hat{\varphi}_{n0}$ the second term in the right side of Eq. (5.14) is proportional to

$$z_{n1} \sim \int\limits_{\Omega}\mathrm{Re}\left[i\sum_{m\neq n}^{N}\varepsilon^{*}_{mk}(r)\varepsilon_{nk}(r)\right] dr \sim$$

$$\sim \sum_{m\neq n}^{N}\int\limits_{\Omega}\mathrm{Re}\left[i\exp\left(i\varphi_n - i\varphi_m\right)\iint\limits_{-\infty}^{\infty}\tilde{\omega}_n(\rho_1 - \rho_n^0)\tilde{\omega}_m(\rho_2 - \rho_m^0)\times\right.$$

$$\left.\times\, \varepsilon(\rho_1)\varepsilon^{*}(\rho_2)G^{*}(r-\rho_1)G(r-\rho_2)d\rho_1 d\rho_2\right] dr.$$

Thus, if 1) the receiving aperture is divided into subapertures of identical form, that is, $\tilde{\omega}_m(\rho) = \tilde{\omega}_n(\rho)$; 2) the centers of the subapertures ρ_n^0 form a periodic grid with the translation vectors C_1 and C_2; 3) the function $\varepsilon(\rho)\exp\left(\frac{ik}{2R}|\rho|^2\right) \equiv \varepsilon'(\rho)$ is real and has the periods C_1 and C_2; and, 4) the region Ω is central symmetric, then for the phase distortions $\varphi_n = \pi n$, we derive

$$z_{n1} \sim \sum_{m\neq n}^{N}\cos\pi(n-m)\mathrm{Re}\left\{i\int\limits_{\Omega}\left[\iint\limits_{-\infty}^{\infty}\tilde{\omega}(\rho_1 - \rho_n^0)\tilde{\omega}(\rho_2 - \rho_m^0)\times\right.\right.$$

$$\times\, \varepsilon(\rho_1)\varepsilon^{*}(\rho_2 + \rho_m^0 - \rho_n^0)\exp\left[-\frac{ik}{2R}|\rho_1|^2 + \frac{ik}{2R}|\rho_2 + \rho_m^0 -$$

$$\left.\left. - \rho_n^0|^2 + \frac{ik}{R} r(\rho_1 - \rho_2)\right] d\rho_1 d\rho_2\right]\times$$

$$\left.\times \exp\left(i\frac{k}{R} r(\rho_n^0 - \rho_m^0)\right)dr\right\} \sim \sum_{m\neq n}^{N}\cos\pi(n-m)\times$$

$$\times\, \mathrm{Re}\left\{\int\limits_{\Omega} dr\left|\int\limits_{-\infty}^{\infty}\tilde{\omega}(\rho - \rho_n^0)\varepsilon'(\rho)\exp\left(i\frac{k}{R} r\rho\right)d\rho\right|^2\times\right.$$

$$\left.\times \sin\left[\frac{k}{R} r(\rho_n^0 - \rho_m^0)\right]\right\} = 0,$$

since with real $\varepsilon'(\rho)$ the integrand represents the product of even and add functions. In this case Eq. (5.14) gives the incorrect estimate $\hat{\varphi}_{n1} = \ldots = \hat{\varphi}_{nk} = 0$. We shall now construct the function $E(r)$ describing the diffuse object that satisfies the conditions (3) and (4). By virtue of the periodicity of $\varepsilon'(\rho)$ the condition

$$\int\limits_{\Omega} E(r)\exp\left[\frac{ik}{2R}\left(|r|^2 - 2r\rho\right)\right]\times$$

$$\times\left[\exp\left(\frac{ik}{R} r\left(C_1 n_1 + C_2 n_2\right)\right) - 1\right]dr = 0$$

should be satisfied, that is, $E(r)$ can be taken as a collection of bright points with the coordinates r_{n_1,n_2,n_3} for which $r_{n_1,n_2,n_3}\,(C_1 n_1 + C_2 n_2) = \lambda R n_3$ (n_1, n_2 and n_3 are integers). From the above condition (4), all the points should lie within an arbitrary central symmetric region Ω. The real character of $\varepsilon'(\rho)$ imposes the conditions on the amplitudes $A(-r) = A(r)$ and the phases $\alpha(-r) = -\alpha(r) - \frac{\kappa}{R}|r|^2$ of individual bright points. We thus construct a class of diffuse objects for which the usage of relation (5.14) is inefficient even with the available *a priori* information about the form of the function $u(r)$.

Based on formula (5.14), the adaptation by means of a sharpness function

$$S_{17} = \int_{\Omega} V\left(-\frac{R}{z}x\right)\left|\int_{-\infty}^{\infty}\omega(\rho)\varepsilon(\rho)G(x-\rho)d\rho\right|^2 dx \qquad (5.15)$$

was proposed, where $V(-\frac{R}{z}x)$ is the image of the mask $V(r)$. It is easy to see that Eq. (5.15) is the generalization of Eq. (3.8) in the presence of noise for finite time of recording, and as N_0 — 0, T—∞ and $V(-\frac{R}{z}x)$ — $u(-\frac{R}{z}x)$. The conclusions drawn for S_4 are therefore valid for S_{17}.

Thus, despite the great scientific importance of Refs. 29 and 40–42 dealing with the application of the adaptive Bayes approach to the problem of measurements of and compensation for the atmospheric phase distortions, the usage of the obtained results in real adaptive systems without *a priori* information is somewhat limited.

6. ADAPTIVE ADJUSTMENT OF AN OUTGOING WAVE

As it was shown earlier, the radiation scattered by unknown extended object is not applicable to measure atmospheric phase fluctuations, since it is impossible to separate the phase distribution of the object field against the background of the phase fluctuations themselves. It may therefore be desirable that a well–known reference surface (e.g., point) be artificially constructed on the object and then Used to compensate for the object field distortions.

Multiple re–emission can be used for adaptive formation of a reference point ("beacon") on the object.[8] In this case, the transmitting aperture each time re–emits the field which is complex conjugate to the received field. After the first re–emission on a specular object (or on the object with specular zones) the field is formed

$$E'(r) = \int_{\Omega}\int_{-\infty}^{\infty}\omega(\rho)E^{*}(r_1)G(r_1-\rho)G(r-\rho)\,dr_1 d\rho \sim$$

$$\sim \int_{\Omega} E^{*}(r_1)g_0(r_1-r)dr_1 \sim E^{*}(r),$$

and the field proportional to $|E(r)|^{2n}$ is formed after n cycles. If there is the brightest zone on the object, then a reference point is formed on it after several cycles. If there are some equally bright zones on that object, the region of radiation focusing is determined by fluctuations.

This procedure exhibits significant disadvantages. It is intended to operate with the objects having fairly "bright" specular zones. If there are some of them and their "brightness" is comparable, the number of cycles n Increases sharply, that would be time–consuming. There are no specular zones on the diffuse object, and an infinite number of re–reflections is required to concentrate the radiation in the zone with maximum reflectance.

To form a reference "point" on the object, it was suggested in many of the papers that the outgoing wave be preliminary adaptively processed in order to compensate for the distortions of the wave caused by in the atmosphere in the process of wave propagating to the object. If the wavefront distribution at the exit from the illuminating aperture is known (e.g., a plane front), then the adaptive formation of a "bright" speckle on the object is indicative of the fact that the total phase distribution, after the wave has passed through the adaptive element and the atmosphere, obeys the condition

$\psi(\rho) = a + b\rho.$

When the reflected field is received conventionally through a fixed adaptive element with preliminary illumination of the object, the atmospheric phase distortions are to be compensated and the image can be constructed. The complexity of this approach consists in the formation of the given "point" or speckle based on some criterion. To do this, the sharpness are also employed.

The authors of Ref. 29 on the basis of the functions (3.8) or (5.15) suggest to use as a mask $M(x)$ a circular slit whose diameter is approximately equal to the dimension of the image of the "bright" speckle which may be formed on the object at a point r_0

$$S_{18} = \int_{\Omega_p}\tilde{\omega}(x)\left|\int_{-\infty}^{\infty}\omega(\rho)\int_{-\infty}^{\infty}\omega_0(\rho')\varepsilon(\rho')\,\times\right.$$

$$\left.\times\,\exp[i\psi(\rho)+i\psi(\rho')]\;G(\rho-x)d\rho'd\rho\right|^2 dx, \qquad (6.1)$$

where

$$\omega_0(\rho') = \begin{cases} 1, & \rho'\in\Omega_{tr}, \\ 0, & \rho'\notin\Omega_{tr} \end{cases}$$

and

$$\omega(x) = \begin{cases} 1, & x\in g_0(-R/z\cdot x_0), \\ 0, & x\notin g_0(-R/z\cdot x_0). \end{cases}$$

The outgoing field is naturally assumed to be coherent in time. Taking into account that $\omega(x)$ — $\delta(-R/z\cdot x_0)$, it can realy be shown that the product of integrals In Eq. (6.1) over Ω and Ω_p

$$S_{18} = \frac{1}{2}(\lambda^2 Rz)^{-2}\iint_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\exp[i\psi(\rho_1)-i\psi(\rho_2)]\times$$

$$\times\int_{\Omega}\int_{\Omega}E(r_1)E^{*}(r_2)z(r_1)z^{*}(r_2)G(r_1-\rho_1)\times$$

$$\times\,G^{*}(r_2-\rho_2)dr_1dr_2\int_{\Omega_p}\omega(x)G(\rho_1-x)G^{*}(\rho_2-x)dx\,d\rho_1 d\rho_2, \qquad (6.2)$$

where

$$z(r) = \frac{1}{\lambda R}\int_{-\infty}\omega_0(\rho')\exp[i\psi(\rho')]G(\rho'-r)d\rho' \qquad (6.3)$$

is real, when

$$E(r_1)Z(r_1)E^{*}(r_2)Z^{*}(r_2) \quad g_0(r-r_0) \quad \delta(r-r_0),$$

that is, when $\psi(\rho') = \text{const}$. As a result, S_{18} is maximized. If the object is diffuse and it is possible to average the accumulated result over its microstructure by means of adaptation, then in Eq. (6.2) by virtue of Eq. (2.8) we obtain

$$\int_\Omega u(r)\,|Z(r)G(r-\rho_1)G^*(r-\rho_2)|^2 dr$$

rather than a double integral over Ω. This does not actually change the situation. Thus the adaptive focusing of an outgoing wave on the object results in the construction of the object image.

As mentioned earlier, the recorded intensity of the reflected field can be averaged over the microstructure of the objects with rough surfaces. Such an averaging is desirable for the reason that in the process of adaptation of the outgoing wave, the function $Z(r)$ illuminates different zones of the object surface. This results in a random change of the speckle-structure of the reflected field (that is, its changes depend not only on exp $[i\psi(\rho')]$, but also on the object surface itself). However, the results of the intensity measurement cannot always be averaged over the object because of the limited time (time during which the atmosphere can be considered "frozen"). The authors of Refs. 45–47 suggested to use the speckle-structure of the field as a criterion for the adaptive processing. As was noted in Refs. 45 and 47, such criteria can be employed if the speckle–structure formed due to the rough surface can separated from the speckle-structure caused by a turbulent atmosphere. When the condition of the object isoplanatism (2.3) is satisfied, in the plane of the entrance (or exit) pupil of the optical system we shall record the intensity

$$I(\rho) = |\varepsilon(\rho)|^2 = \frac{1}{(\lambda R)^2}\left|\int_\Omega E(r)Z(r)G(r-\rho)dr\right|^2, \qquad (6.4)$$

the statistical characteristics of which with condition (2.8), will be equal to

$$\langle I(\rho)\rangle = \frac{1}{(\lambda R)^2}\int_\Omega u(r)\,|Z(r)|^2 dr;$$

$$\sigma_I^2 = \frac{1}{(\lambda R)^4}\left|\int_\Omega u(r)\,|Z(r)|^2 dr\right|^2, \qquad (6.5)$$

where $\langle I(\rho)\rangle$ is the average value and σ_I^2 is the variance of $I(\rho)$.

It becomes apparent from Eq. (6.5) that by taking either $\langle I(\rho)\rangle$ or σ_I^2 (e.g., σ_I^2) as a sharpness function

$$S_{19} = \left|\int_\Omega u(r)\,|Z(r)|^2 dr\right|^2 \qquad (6.6)$$

or

$$S_{19} = \left|\int_\Omega u(r)\left|\int_{-\infty}^{\infty}\omega_0(\rho')e^{i\psi(\rho')}G(\rho'-r)d\rho'\right|^2 dr\right|^2,$$

we see that it reaches maximum for $u(r) = |Z(r)|^2$, that is, when the total light energy is incident on the object. Taking account of the fact that a significant contribution to the formation of the image is made by the phase,[13] it is easy to see that $\psi(\rho')$ satisfies the condition

$$\psi(\rho') + \arg\int_\Omega u(r)G(r-\rho)dr = 0, \qquad (6.7)$$

that is, there no compensation for atmospheric distortions. The other statistical criterion is the spatial correlation length ρ_k of the intensity $I(\rho)$ or the average dimension of the speckle in the intensity distribution. Taking this criterion as a sharpness function, we obtain

$$S_{20} = \frac{\left[\int_\Omega u^2(r)\,|Z(r)|^4 dr\right]^{1/2}}{\int_\Omega u(r)\,|Z(r)|^2 dr} \qquad (6.8)$$

following Ref. 45.

Since in Eq. (6.8) the numerator is proportional to $S^{1/2}$ and the denominator is proportional to S, S_{20} increases with decrease of Ω. That is possible only when the image formation zone $Z(r)$ reduces (due to focusing). Thus, S_{20} is at a maximum when

$$\int_{-\infty}^{\infty}\omega_0(\rho)e^{i\psi(\rho)}G(\rho-r)d\rho = g_0(r-r_0),$$

and the width of g_0 is much smaller than Ω. This implies the condition of compensation for atmospheric distortions $\psi(\rho') = \text{const}$ is satisfied.

The so–called interference criteria (sharpness functions) were suggested in Refs. 45 and 46, which describe the interaction between the outgoing and reflected waves

$$S_{21} = \left|\int_{-\infty}^{\infty}\omega(\rho)\varepsilon^0(\rho)\varepsilon^*(\rho)e^{i\psi(\rho)}d\rho\right|^2, \qquad (6.9)$$

where $\varepsilon_0(\rho)$ is the outgoing wave, $\omega(\rho) \equiv \omega_0(\rho)$ (the receiving and transmitting apertures are identical). If any information about the object is valuable S_{21} converts to the iteration algorithm (5.11). In one limiting case (when there is comprehensive information about, the object $E(r)$) S_{21} is equivalent to S_{14} while in the other (when there are no information) S_{21} is equivalent to S_{16}. The same results might be obtained, when Re J or Im J (J is the integral over $d\rho$ in Eq. (6.9)) are used in expression for S_{21} instead of the square modulus.

The so–called spectral criterion

$$I_3 = \iint^{\infty}\omega(\rho_1)\omega(\rho_2)\varepsilon^0(\rho_1)\varepsilon^*(\rho_2)\exp[i\psi(\rho_1) - i\psi(\rho_2)] \times$$
$$\times\, \tilde{M}(\rho_1-\rho_2)d\rho_1 d\rho_2 = \int F^0(x)F^*(x)m(x)dx = \text{const}$$

was proposed as a generalization of Eq. (6.9) In Refs. 7–9, where

$$F^0(x)=F\left\{\omega(\rho)\varepsilon^0(\rho)\exp(i\psi(\rho))\right\},$$

$$F(x) = F\left\{\omega(\rho)\varepsilon(\rho)\exp(i\psi(\rho)\right\},$$

F is the Fourier transform operator, $\tilde{M}(\rho_1 - \rho_2)$ is the certain weighting function, and $m(x)$ is the Fourier transform of $\tilde{M}(\rho_1 - \rho_2)$. Since for the unknown object neither $\tilde{M}(\rho_1 - \rho_2)$, no $m(x)$ are known, the approximations $\tilde{M}(\rho_1 - \rho_2) = \delta(\rho_1 - \rho_2)$ and $m(x) = 1$ are most often used. Taking into account that $F^0(x) \sim E^0(x)$ and $F(x) \sim E(x)$, we can construct a spectral sharpness function in the form

$$S_{22} = \int\limits_{\Omega_p} E^0(x)E^*(x)dx. \tag{6.11}$$

It can readily be shown that

$$S_{22}= \frac{1}{2}(\lambda^2 Rz)^{-2}\iint\limits_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\exp[i\psi(\rho_1) - i\psi(\rho_2)]\times$$

$$\times \int\limits_{\Omega} E^*(r)\exp(-ik/Rr\rho_1)dr \times$$

$$\times \int\limits_{\Omega_p} E^0(x)\exp(ik/zx\rho_2)dxd\rho_1 d\rho_2 , \tag{6.12}$$

from which it appears that the condition of compensation for atmospheric distortions is satisfied either for identical $E^0(x)$ and $E(r)$, which is similar to the *a priori* knowledge of $E(r)$, or for central symmetry of $E^0(x)$ and $E(r)$, which restricts the class of the objects under study (specular objects). The modifications of S_{22} are possible, namely, the recursion algorithm S_{23}, which is similar to Eqs. (3.8), (4.2), and (5.12)

$$(S_{23})_k = \int\limits_{\Omega_p} E_k^0(x)E_k^*(x)dx, \tag{6.13}$$

where

$$E_k^0(x) = E_{k-1}(x).$$

If the object is diffuse, then as it has been mentioned above, in the case of phase modulation of the outgoing wave the adaptation is harmfully affected by random change of the field speckle structure. For this reason the authors of Refs. 7–9 suggest to use instead of S_{21} ... S_{23} their values averaged over the ensemble of realizations

$$\langle S_{21}\rangle= \iint\limits_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\varepsilon^0(\rho_1)\varepsilon^{0*}(\rho_2)\langle\varepsilon^*(\rho_1)\varepsilon(\rho_2)\rangle$$

$$\times \exp[i\psi(\rho_1) - i\psi(\rho_2)]d\rho_1 d\rho_2$$

and

$$\langle|S_{22}|^2\rangle= \int\limits_{\Omega_p}\int\limits_{\Omega_p} E(x_1)E_0^*(x_2)\langle E^*(x_1)E(x_2)\rangle dx_1 dx_2. \tag{6.15}$$

It can readily be shown that

$$S_{26} = \frac{1}{2}(\lambda^2 Rz)^{-2}\iint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\exp[i\psi(\rho_1) - i\psi(\rho_2)]$$

$$\times \int\limits_{\Omega_p}\int\limits_{\Omega_p} E^0(x_1)E\ \ (x_2)\exp[ik/z\cdot(x_1\rho_1 - x_2\rho_2)]dx_1 dx_2 \times$$

$$\times \int\limits_{\Omega} u(r)\ \exp\ [ik/Rr(\rho_1 - \rho_2)]\ dr\ d\rho_1 d\rho_2, \tag{6.16}$$

and its maximization resulting in compensation for the atmospheric distortions $\psi(\rho) = \text{const}$ is achieved provided that

$$\arg\ \varepsilon_1^0(\rho) - \arg\ \varepsilon^0(\rho) - \arg\int\limits_{\Omega} u(r)\exp(ik/R\cdot r\rho)dr = 0,$$

that is, when *a priori* information about the object shape is available. It is obvious that the sharpness function S_{24} is completely equivalent to S_4.

The formation of an artificial "reference" zone on an object is quite helpful in distinguishing between the phase distribution of the object field and the phase distortions, when this "reference" zone is almost independent of the object shape. That is why we attempt to focus an outgoing wave into a "bright" speckle, whose width is much smaller than Ω. Yet, there are some limitations determined by possible divergence of the outgoing wave and the distance to the object that complicates the formation of a reference zone on the surface of the remote object.

The "reference" zone can be constructed in a different way. Under conditions of the frozen atmosphere the object is successively illuminated by the field with the known distribution of the complex amplitude $\varepsilon^1(\rho)$ and the conventional plane wave $\varepsilon^0(\rho)$ (Ref. 48). When there are no atmospheric distortions in the image plane, we derive successively $|E(x)|^2 |Z_1(x)|^2$ and $|E(x)|^2$, where

$$Z^1(x) = \frac{1}{\lambda z}\int\limits_{-\infty}^{\infty}\omega(\rho)\varepsilon^1(\rho)G(\rho - x)d\rho.$$

The quotient of division is $|Z^1(x)|^2$ and does not depend on the object shape. Thus, the function $|Z^1(x)|^2$ can serve as a criterion of the adaptive compensation for distortions. The corresponding sharpness function has the form

$$S_{27} = \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2 + n_0} \times$$

$$\times \frac{\left|\int\limits_{\Omega} E(r)Z_{\psi}^1(r)g_1\left(\frac{r}{R} + \frac{x}{z}\right)dr\right|^2 + n_0}{\left|\int\limits_{\Omega} E(r)g_1\left(\frac{r}{R} + \frac{x}{z}\right)dr\right|^2 + n_0}\,dx, \tag{6.17}$$

where

$$Z^1_\psi(r) = \frac{1}{\lambda R}\int_{-\infty}^{\infty}\omega(\rho)\varepsilon^1(\rho)\exp[i\psi(\rho)]G(\rho - r)d\rho.$$

Because

$$g_1\left[\frac{r}{R} + \frac{x}{z}\right] = \frac{1}{S_0}\int_{-\infty}^{\infty}\omega(\rho)\exp\left[i\psi(\rho) + \right.$$

$$\left. + ik\left[\frac{r}{R} + \frac{x}{z}\right]\rho\right]d\rho,$$

then with condition of low–level noise n_0 the absolute maximum can be achieved when the second term under the integral over dx is equal to $|Z_1(x)|^2$ which is possible when $\psi(\rho)$ = const, that is, with compensation for atmospheric distortions. Thus, the sharpness function is universal for coherent illumination of the object and makes it possible to compensate for the atmospheric distortions when an a priori Information about the object shape is not available.

7. METHODS FOR FINDING THE MAXIMUM OF THE SHARPNESS FUNCTION

It was demonstrated in the foregoing sections that the problem of the undistorted image formation is reduced to the realization of the absolute maximum of the sharpness functions enumerated above. The simplest way to find the absolute maximum consists of successive sorting of all the possible states of the adaptive element. The number of these states, however, is too large. For instance, if the element incorporates ten individual subapertures, and each of these subapertures can occupy ten different positions (shifted from zero to 2π with the increment π/5) then the total number of possible states is 10^{10}. Then for the real–time operation of the adaptive element, it is necessary to measure the reflected signal intensity during time smaller than 10^{-10} $t_f \sim 10^{-13}$ s, where $t_f \sim 10^{-3}$ s is time during which the atmosphere can be considered frozen. Such measurements are unreal as far as the energetic and speed of response are concerned. This methods would call for the use of the other methods in order to seek the absolute maximum more rapidly.[26–28] In the process, however, the problem of secondary extrema arises. Martin[49] demonstrated that in the case of phase distortions being above 0.357 π along the aperture (when observations are performed through the atmosphere, this condition is practically always satisfied) the functional S_1 has the secondary maxima. If the sharpness function has the secondary extreme, then while seeking its maxima, we can find not the absolute but local maximum. Since there are no a priori information about the object, it is impossible to distinguish which maximum has been found absolute or local one, we cannot determine with reliability whether the atmospheric phase distortions have been compensated using an adaptive system or not. Therefore, if the sharpness function has secondary extrema, the image reconstruction turns out to be problematic.

The simplest among the existing ways to seek the maximum[22,50] are given below.

1) Different phase perturbations are successively brought in one of the subapertures with unchanged positions of the rest of the subapertures then the optimal phase shift providing for maximum of the sharpness function is found and stored for it. This subaperture is further returned to zero position, and after sampling all the apertures they are simultaneously adjusted in an optimal way.

2) After the optimal position has been found, each of the subaperture preserves in it, but does not come back to the initial position. The optimal position of the last subaperture must provide for the absolute maximum of the sharpness function.

Let us now examine whether the absolute maximum of the considered sharpness functions is achievable when using these two approaches. The adaptive element is assumed to represent a collection of movable subapertures Δ_n (n = 1,..., N and N is the number of subapertures) with the identical area A and with the rectangular response, that is, a change of the phase due to adaptation

$$\theta(\rho) = \theta_n\tilde{\omega}(\rho - \rho^0_n) = \begin{cases}\theta_n, & \rho \in \Delta_n,\\ 0, & \rho \notin \Delta_n,\end{cases} \quad (7.1)$$

where θ_n is a change of the phase caused by the displacement of the subaperture Δ_n.

1) For the displacement of the subaperture A , which produces a change of the phase variation в , the functional increment ΔS_1 has the form

$$\Delta S_1(\theta_n) = S_1(\theta_n) - S_1(0) = 2\int_{\Omega_p} I(x)\Delta I(x)dx, \quad (7.2)$$

where $S_1(\theta_n)$ and $S_1(0)$ are the sharpness functions for changed and undisplaced subapertures Δ_n and

$$\Delta I(x) = \frac{1}{2}(\lambda^2Rz)^{-2}\int_\Omega u(r)dr\iint_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\times$$

$$\times\left\{\exp[i\theta(\rho_1) - \theta(\rho_2)] - 1\right\}\exp\left[i\varphi(\rho_1) - i\varphi(\rho_2) - \right.$$

$$\left. - ik(\rho_1 - \rho_2)\left[\frac{r}{R} + \frac{x}{z}\right]\right]d\rho_1d\rho_2. \quad (7.3)$$

Substituting Eqs. (2.9) and (7.3) into Eq. (7.2) and integrating over the variable x, we find

$$\Delta S_1(\theta_n) \sim \iiint_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_1 - \rho_2 + \rho_3)\times$$

$$\times\exp[i\varphi(\rho_1) - i\varphi(\rho_2) + i\varphi(\rho_3) - i\varphi(\rho_1 - \rho_2 + \rho_3)]\times$$

$$\times\{\exp[i\theta(\rho_1) - i\theta(\rho_2)] - 1\}\left|\int_\Omega u(r)\times\right.$$

$$\left.\times\exp\left[ik\frac{r}{R}(\rho_2 - \rho_1)\right]dr\right|^2 d\rho_1d\rho_2d\rho_3. \quad (7.4)$$

Employing the notation

$$\alpha_n e^{i\beta_n} = \int_{-\infty}^{\infty}\omega_n(\rho)e^{i\varphi(\rho)}C(\rho - \rho^0_n)d\rho \quad (7.5)$$

and

$$C(\rho - \rho_n^0) = \left|\int_\Omega u(r)\exp\left[ik\frac{r}{R}(\rho - \rho_n^0)\right]dr\right|^2 \times$$

$$\times \int_{-\infty}^{\infty}\omega(\rho_3)\omega(\rho - \rho_n^0 + \rho_3) \times$$

$$\times \exp[i\varphi(\rho_3) - i\varphi(\rho - \rho_n^0 + \rho_3)]d\rho_3, \tag{7.6}$$

where $\omega_n(\rho)$ is the aperture function which is equal to zero on the subaperture Δ_n and unity on the remaining part of the aperture and ρ_n^0 is the coordinate of the center of the subaperture A , we finally derive

$$\Delta S_1(\theta_n) \sim -\alpha_n \Delta\sin\frac{\theta_n}{2}\sin\left[\varphi(\rho_n^0) + \frac{\theta_n}{2} - \beta_n\right] \sim$$

$$\sim \alpha_n\Delta\left\{\cos[\varphi(\rho_n^0) + \theta_n - \beta_n] - \cos[\varphi(\rho_n^0) - \beta_n]\right\}. \tag{7.7}$$

From the measurements of $\Delta S_1(\theta_n)$ with different phase increments θ_n it is possible to find the phase difference $\varphi(\rho_n^0) - \beta_n$ for each of the subapertures Δ_n from Eq. (7.7). The value β_n is a function of n, and therefore Eq. (7.7) does not enable one to measure the phase distortions and to reconstruct the object front since it is impossible to calculate β_n without the knowledge of phase distortions. The condition of maximization of the sharpness function S_1 by such a method of seeking

$$d\Delta S_1(\theta_n)/\,d\theta_n = 0 \tag{7.8}$$

for all n leads to the equality

$$\varphi(\rho_n^0) + \theta_n - \beta_n = 0. \tag{7.9}$$

The fact that this equality is satisfied does not provide for the wavefront reconstruction and the undistorted image restoration. Maximization of the sharpness function S_4, where the undistorted image $I_0(x)$ is used as a mask, leads to the following expression:

$$\Delta S_4(\theta_n) \sim \alpha_n^1\Delta\left\{\cos\left[\varphi(\rho_n^0) + \theta_n - \beta_n^1\right] - \right.$$

$$\left. - \cos\,[\varphi(\rho_n^0) - \beta_n^1]\right\}, \tag{7.10}$$

where

$$\alpha_n^1 e^{i\beta_n^1} = \int_{\infty}^{\infty}\omega_n(\rho)e^{i\varphi(\rho)}\left|\int_\Omega u(r)\exp[i\frac{k}{R}(\rho - \rho_n^0)]dr\right|^2 d\rho.$$

In the general case the coefficients β_n^1 with different n are not equal, and maximization of the functional S_4 does not provide for the image restoration. For the point object placed at the origin of the coordinates ($u(r) = \delta(r)$)

$$\alpha_n^1 e^{i\beta_n^1} = \int_{-\infty}^{\infty}\omega(\rho)e^{i\varphi(\rho)}d\rho - \Delta e^{i\varphi(\rho_n^0)},$$

and with the number of subapertures $N \gg 1$ with high probability $\beta_n^1 \approx \arg\int_{-\infty}^{\infty}\omega(\rho)\,e^{i\varphi(\rho)}\,d\rho$ because of the smallness of the second term, that is, the coefficients β_n^1 are approximately identical. Thus, this method of seeking the absolute maximum of the sharpness functions S_4 and S_8, even when an *a priori* information about the extended object shape is available, yields the secondary maxima. Finding the absolute maximum with the help of S_4 and S_8 is possible only for the point objects. In the case of a point object the maximization of the sharpness function S_{10}, whose mask coincides with the spatial spectrum of the undistorted image $F_0(f)$, gives

$$\Delta S_{10}(\theta_n) \sim \mathrm{Re}\left\{\int_{\Omega_f} M_0(f)H^*(f)df\int_{\Omega_f} H_0^*(f)\Delta H(f)df\right\}, \tag{7.11}$$

where the atmosphere-lens transfer function is

$$H(f) = S^{-1}\int_{-\infty}^{\infty}\omega(\rho)\omega\left(\rho - 2\pi\frac{z}{k}f\right) \times$$

$$\times \exp\left[i\varphi(\rho) - i\varphi\left(\rho - 2\pi\frac{z}{k}f\right)\right]d\rho,$$

$$H_0(f) = H(f)\Big|_{\varphi(\rho)=0}$$

and

$$\Delta H(f) \sim \Delta\sin\frac{\theta_n}{2}\left\{\omega\left(\rho_n^0 - 2\pi\frac{z}{k}f\right) \times\right.$$

$$\times \exp\left[i\varphi(\rho_n^0) - i\varphi\left(\rho_n^0 - 2\pi\frac{z}{k}f\right) + \frac{i\theta_n}{2}\right] -$$

$$- \omega\left(\rho_n^0 + 2\pi\frac{z}{k}f\right)\exp\left[i\varphi(\rho_n^0) + 2\pi\frac{z}{k}f\right) -$$

$$\left. - i\varphi(\rho_n^0) - \frac{i\theta_n}{2}\right]\Big]\Big\}.$$

For the central subapertures Δ_n when

$$\omega\left(\rho_n^0 - 2\pi\frac{z}{k}f\right) = \omega\left(\rho_n^0 + 2\pi\frac{z}{k}f\right) = 1 \text{ and}$$

$$\Delta H(f) \sim \Delta\sin\frac{\theta_n}{2}\sin\left[\varphi(\rho_n^0) - \frac{1}{2}\varphi\left(\rho_n^0 + 2\pi\frac{z}{k}f\right) - \right.$$

$$\left. - \frac{1}{2}\varphi\left(\rho_n^0 - 2\pi\frac{z}{k}f\right) + \frac{\theta_n}{2}\right],$$

the maximization condition

$$d\Delta S_{10}(\theta_n)/d\theta_n \sim \Delta \int_{\Omega_f} H_0^*(f)\sin\left[\theta_n + \varphi(\rho_n^0) - \frac{1}{2}\varphi\left(\rho_n^0 + 2\pi\frac{z}{k}f\right) - \frac{1}{2}\varphi\left(\rho_n^0 - 2\pi\frac{z}{k}f\right)\right]df = 0 \quad (7.12)$$

in general does not meet the requirements on compensation for the atmospheric phase distortions, therefore it is of low probability that the extended object image can be restored with the help of the sharpness functions S_{10} and S_{11}.

For the sharpness function S_9, we derive

$$\Delta S_9 \sim \left.\frac{\Delta I_x(x)I_y(x) - \Delta I_y(x)I_x(x)}{A^2I_y^2(x)}\right|_{x=0} = \frac{2i\Delta\sin\frac{\theta_n}{2}}{AI_y^2(x)}\alpha_n^y\sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^y\right] - \left.\frac{2i\Delta\sin\frac{\theta_n}{2}}{AI_y^2(x)}\alpha_n^x I(x)\sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^x\right]\right|_{x=0}, \quad (7.13)$$

where

$$\alpha_n^z e^{i\beta_n^z} = \int_{-\infty}^{\infty}\omega_n(\rho)C_z(\rho - \rho_n^0) \times \left.\exp\left[i\varphi(\rho) + i\frac{k}{z}x(\rho - \rho_n^0)\right]d\rho\right|_{x=0}$$

$$C_z(\rho - \rho_n^0) = \int_\Omega u_{zz}(r)G(r - \rho)G^*(r - \rho_n^0)dr,$$

and z stands for x or y. Even under the condition

$$\alpha_n^x I_x(0) \sim \alpha_n^y I_y(0) \sim \tilde{C}_n$$

we derive

$$\Delta S_9 \sim 2i\Delta\tilde{C}_n\sin\frac{\theta_n}{2}\cos\left[\frac{\beta_n^x - \beta_n^y}{2}\right] \times \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \frac{\beta_n^x - \beta_n^y}{2}\right]. \quad (7.14)$$

Because β_n^x and β_n^y depend on the concrete subaperture, the atmospheric distortions cannot be compensated successively. Below we examine the sharpness function S_{12}

$$\Delta S_{12}(\theta_n) \sim |F_0(f_0)\Delta H(f_0, \theta_n)|^2 = |F_0(f_0)|^2S_0^{-2} \times \left|\int_{-\infty}^{\infty}\omega(\rho)\omega(\rho - 2\pi\frac{z}{k}f_0)\exp\left[i\varphi(\rho) - i\varphi(\rho - 2\pi\frac{z}{k}f_0)\right] \times \left\{\exp\left\{i\theta_n\left[\tilde{\omega}(\rho - \rho_n^0) - \tilde{\omega}_n(\rho - \rho_n^0 - 2\pi\frac{z}{k}f_0)\right]\right\} - 1\right\}d\rho\right|^2. \quad (7.15)$$

Since we are most interested in the frequencies $0 < |f_0| < \frac{\kappa\rho_0}{2\pi z}$ for which the values $2\pi z/\kappa|f_0|$ exceed a linear dimension of individual subaperture the zones where $\omega(\rho)$ and $\omega(\rho - 2\pi z/\kappa f_0)$ are nonzero do not superimposed. Then

$$\Delta S_{12}(\theta_n) \sim |F_0(f_0)|^2S_0^{-2}\left|\int_{-\infty}^{\infty}\tilde{\omega}(\rho - \rho_n^0)\left\{\omega\left(\rho - 2\pi\frac{z}{k}f_0\right) \times \exp\left[i\varphi(\rho) - i\varphi\left(\rho - 2\pi\frac{z}{k}f_0\right)\right]\left(e^{i\theta_n} - 1\right) \times \omega\left(\rho + 2\pi\frac{z}{k}f_0\right)\exp\left[i\varphi\left(\rho + 2\pi\frac{z}{k}f_0\right) - i\varphi(\rho)\right] \times \left(e^{-i\theta_n} - 1\right)\right\}d\rho\right|^2 = 4\Delta|F_0(f_0)|^2S_0^{-2}\sin\left(\frac{\theta_n}{2}\right) \times \left|\int_{-\infty}^{\infty}\tilde{\omega}(\rho - \rho_n^0)\left\{\omega\left(\rho - 2\pi\frac{z}{k}f_0\right)\exp\left[i\varphi(\rho) - i\varphi\left(\rho - 2\pi\frac{z}{k}f_0\right) + i\frac{\theta_n}{2}\right] - \omega\left(\rho + 2\pi\frac{z}{k}f_0\right) \times \exp\left[i\varphi\left(\rho + 2\pi\frac{z}{k}f_0\right) - i\varphi(\rho) - i\frac{\theta_n}{2}\right]d\rho\right|^2. \quad (7.16)$$

Given that $\omega\left(\rho - 2\pi\frac{z}{k}f_0\right) = \omega\left(\rho + 2\pi\frac{z}{k}f_0\right)$, we finally derive

$$\Delta S_{12} \sim 4\Delta S_0^{-1}|F_0(f_0)|^2\{\cos[S_n(f_0) + \theta_n] - \cos S_n(f_0)\}^2, \quad (7.17)$$

where

$$S_n(f_0) = \varphi(\rho_n^0) - \frac{1}{2}\left[\varphi\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right) + \varphi\left(\rho_n^0 + 2\pi\frac{z}{k}f_0\right)\right]. \quad (7.18)$$

In the physical sense, the value $S_n(f_0)$ indicates the difference of the differences between atmospheric phase distortions at the adjacent points of the receiving aperture. When $|f_0| < \frac{\kappa\rho_0}{2\pi z}$, the value $S_n(f_0) \approx -\frac{1}{2}\left(2\pi\frac{z}{\kappa}|f_0|\right)^{-2}\varphi''_{f_0}(\rho_n^0)$, that is, it is proportional to the second derivative of the function $\varphi(\rho)$ at the point ρ_n^0 in the direction of the vector f_0. The function $\varphi(\rho)$ can be found to within a linear tilt (that is, to within the condition (3.5)) based on the secondary derivatives in the two different directions.

We can write for S_{13}:

$$\Delta S_{13} \sim \left| \iint\limits_{\Omega_f} \frac{\Delta I_2(x)I(x) - I_2(x)\Delta I(x)}{I^2(x)} \exp(-2\pi i f_0 x)dx \right|^2, \quad (7.19)$$

where

$$\Delta I_2(x) = \frac{1}{2}(\lambda^2 Rz)^{-2} \iint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)C_2(\rho_1)C_2^*(\rho_2) \times$$
$$\times \exp[i\varphi(\rho_1) - i\varphi(\rho_2)]\exp\left[i\frac{k}{z}x(\rho_1 - \rho_2)\right] \times$$
$$\times \left[\tilde{\omega}(\rho_1 - \rho_n^0)\exp\left(i\frac{\theta_n}{2}\right) - \tilde{\omega}(\rho_2 - \rho_n^0) \times\right.$$
$$\left.\times \exp\left(-i\frac{\theta_n}{2}\right)\right]d\rho_1 d\rho_2 \cdot 2i \sin\frac{\theta_n}{2},$$

$$\Delta I(x) = \frac{1}{2}(\lambda^2 Rz)^{-2} \iint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)C(\rho_1)C^*(\rho_2) \times$$
$$\times \exp\left[i\varphi(\rho_1) - i\varphi(\rho_2) + i\frac{k}{z}x(\rho_1 - \rho_2)\right] \times$$
$$\times \left[\tilde{\omega}(\rho_1 - \rho_n^0)\exp\left(i\frac{\theta_n}{2}\right) - \tilde{\omega}(\rho_2 - \rho_n^0) \times\right.$$
$$\left.\times \exp\left(-i\frac{\theta_n}{2}\right)\right]d\rho_1 d\rho_2 \cdot 2i \sin\frac{\theta_n}{2},$$

$$C(\rho) = \int\limits_{\Omega} E(r)G(r - \rho)dr,$$

and

$$C_2(\rho) = \frac{1}{2}\int\limits_{\Omega} E(r)\left[\exp(i\varphi_1 + i\pi f_0 r) + \exp(i\varphi_2 - i\pi f_0 r)\right] \times$$
$$\times G(r - \rho)dr = \frac{1}{2}\varepsilon_1\left(\rho - 2\pi\frac{z}{k}f_0\right) +$$
$$+ \frac{1}{2}\varepsilon_1\left(\rho + 2\pi\frac{z}{k}f_0\right).$$

By substituting the expressions for $\Delta I(x)$ and $\Delta I_2(x)$ into Eq. (7.19), we derive

$$\Delta S_{13} \sim \left| 2\Delta i \sin\frac{\theta_n}{2} \left[A_2(\rho_n^0)F_1^*(\rho_n^0 - 2\pi\frac{z}{k}f_0) - \right.\right.$$
$$\left. - A_2^*(\rho_n^0)F_1(\rho_n^0 + 2\pi\frac{z}{k}f_0)\right] - 2\Delta i \sin\frac{\theta_n}{2} \times$$
$$\left.\times \left[A(\rho_n^0)F_2^*(\rho_n^0 - 2\pi\frac{z}{k}f_0) - A^*(\rho_n^0)F_2(\rho_n^0 + 2\pi\frac{z}{k}f_0)\right]\right|^2, \quad (7.20)$$

where

$$A_2(\rho_n^0) = \frac{1}{4}\exp\left[i\frac{\theta_n}{2} + i\varphi(\rho_n^0)\right]\left[\varepsilon_1\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right) + \right.$$
$$\left. + \varepsilon_1\left(\rho_n^0 + 2\pi\frac{z}{k}f_0\right)\right]\exp\left(i\frac{k}{z}x\rho_n^0\right)\int\limits_{-\infty}^{\infty}\omega_n(\rho) \times$$
$$\times \left[\varepsilon_1^*\left(\rho - 2\pi\frac{z}{k}f_0\right) + \varepsilon_1^*\left(\rho + 2\pi\frac{z}{k}f_0\right)\right] \times$$
$$\times \exp\left[-i\varphi(\rho) - i\frac{k}{z}x\rho\right]d\rho,$$

$$A(\rho_n^0) = \exp\left[i\frac{\theta_n}{2} + i\varphi(\rho_n^0)\right]\varepsilon(\rho_n^0)\exp\left(i\frac{k}{z}x\rho_n^0\right) \times$$
$$\times \int\limits_{-\infty}^{\infty}\omega_n(\rho)\varepsilon^*(\rho)\exp\left[-i\varphi(\rho) - i\frac{k}{z}x\rho\right]d\rho,$$

$$F_1\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right) = \frac{1}{2}\int\limits_{\Omega_p}\left(\frac{1}{I(x)}\right)\int\limits_{\Omega}E(r)\left[\exp(i\varphi_1 + \right.$$
$$\left. + \pi i f_0 r) + \exp(i\varphi_2 - \pi i f_0 r)\right]g\left(\frac{r}{R} + \frac{x}{z}\right) \times$$
$$\times \exp\left[i\frac{k}{z}x\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)\right]dx,$$

and

$$F_2\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right) = \int\limits_{\Omega_p}\left(\frac{I_2(x)}{I^2(x)}\right) \times$$
$$\times \exp\left[i\frac{k}{z}x\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)\right]dx,$$

so that

$$\Delta S_{13} \sim \left| 2\Delta i \sin\frac{\theta_n}{2}\alpha_n^1 \sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^1\right) - \right.$$
$$\left. - \Delta 2i \sin\frac{\theta_n}{2}\alpha_n^2 \sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^2\right)\right|^2, \quad (7.21)$$

where

$$\alpha_n^1 \exp(i\beta_n^1) = \frac{1}{2}\left[\varepsilon\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right) + \right.$$
$$\left. + \varepsilon\left(\rho_n^0 + 2\pi\frac{z}{k}f_0\right)\right]F_1^*\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)$$

and

$$\alpha_n^2 \exp(i\beta_n^2) = \varepsilon(\rho_n^0)F_2^*\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right).$$

Assuming that $\alpha_n^1 \sim \alpha_n^2 \sim \alpha_n$ since

$$\left|F_1\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)\right|^2 \sim \left|F_2\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)\right|^2$$

and the frequency f_0 is so small that the modulus $|\varepsilon(\rho_n^0)|$ changes insignificantly at a distance of the order of $2\pi z/\kappa|f_0|$, that is,

$$|\varepsilon(\rho_n^0)| \sim \left|\varepsilon\left(\rho_n^0 - 2\pi\frac{z}{k}f_0\right)\right| \sim \left|\varepsilon\left(\rho_n^0 + 2\pi\frac{z}{k}f_0\right)\right| ,$$

we finally derive

$$\Delta S_{13} \sim 16\Delta^2\sin^2\frac{\theta_n}{2}(\alpha_n)^2\sin^2\left(\frac{\beta_n^2 - \beta_n^1}{2}\right) \times \cos\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \frac{\beta_n^1 + \beta_n^2}{2}\right). \quad (7.22)$$

Since β_n^1 and β_n^2 depend simultaneously on the object and on the atmosphere in different ways for each of the subapertures Δ_n, the successive compensation for distortions in each of Δ_n is impossible.

Let us now consider the sharpness function S_{14}:

$$\Delta S_{14} = \iint\limits_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\varepsilon_1(\rho_1)\varepsilon_0^*(\rho_1)\varepsilon_1^*(\rho_2)\varepsilon_0(\rho_2) \times \exp[i\varphi(\rho_1) - i\varphi(\rho_2)]\exp[i\theta(\rho_1) - i\theta(\rho_2) - 1]d\rho_1 d\rho_2. \quad (7.23)$$

For $\varepsilon_0(\rho) = \varepsilon_1(\rho)$ we derive

$$\Delta S_{14} \sim 2i\Delta\sin\frac{\theta_n}{2}|\varepsilon_1(\rho_n^0)|^2\alpha_0\sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_0\right), \quad (7.24)$$

where

$$\alpha_0 e^{i\beta_0} = \int\limits_{-\infty}^{\infty}\omega_n(\rho)|\varepsilon_1(\rho)|^2 e^{i\varphi(\rho)}d\rho.$$

Since for the number of subapertures $N \gg 1$,

$$\beta_0 \approx \arg\int\limits_{-\infty}^{\infty}\omega(\rho)|\varepsilon_1(\rho)|^2 e^{i\varphi(\rho)}d\rho,$$

that is, β_0 is approximately constant for all Δ_n, then the values of the phase $\varphi(\rho_n^0)$ = const can be compensated for each subaperture what is relevant for the image restoration.

For the sharpness function S_{18} we can write

$$\Delta S_{18} = \iiiint\limits_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_4)\exp[i\varphi(\rho_1) - i\varphi(\rho_2) + i\varphi(\rho_3) - i\varphi(\rho_4)]C(\rho_3 - \rho_1)C^*(\rho_4 - \rho_2) \times \exp\left[i\frac{k}{z}x_0(\rho_3 - \rho_4)\right]\left[\exp[i\theta(\rho_1) - i\theta(\rho_2) + i\theta(\rho_3) - i\theta(\rho_4)] - 1\right]d\rho_1 d\rho_2 d\rho_3 d\rho_4,$$

where

$$C(\rho_1 - \rho_j) = \frac{1}{2}(\lambda^2 Rz)^{-2}\int\limits_{\Omega}E(r)G(r - \rho_1 + \rho_j)dr .$$

Taking into account Eq. (7.1), it is possible to write down the expression in the square brackets in the form

$$[\cdot] = \tilde{\omega}(\rho_1 - \rho_n^0)\left[e^{i\theta_n} - 1\right] + \tilde{\omega}(\rho_2 - \rho_n^0)\left[e^{-i\theta_n} - 1\right] + \omega(\rho_2 - \rho_n^0)\left[e^{i\theta_n} - 1\right] + \tilde{\omega}(\rho_4 - \rho_n^0)\left[e^{-i\theta_n} - 1\right],$$

and then

$$\Delta S_{18} \sim 2\Delta i\sin\frac{\theta_n}{2}\left[\alpha_{x_0}\alpha_n^1\sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_{x_0} - \beta_n^1 - \beta_n^0\right) + \alpha_n^2\sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_{x_0} - \beta_n^2 - \beta_n^0\right)\right] + 2i\sin\frac{\theta_n}{2}P\left[\alpha_{x_0}\sin\left(\theta_n + 2\varphi(\rho_n^0) - \beta_{x_0} - \beta_n^0 - \beta_p\right) + \alpha_n^1\sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^1 - \beta_n^0 - \beta_p\right) + \alpha_n^1\sin\left(\frac{\theta_n}{2} + \varphi(\rho_n^0) + \beta_n^1 + \beta_n^0 + \beta_p\right)\right], \quad (7.28)$$

where

$$\alpha_{x_0}\exp\left[i\beta_{x_0}\right] = \iint\limits_{-\infty}^{\infty}\omega_n(\rho_1)\omega_n(\rho_2)\exp[i\varphi(\rho_1) - i\varphi(\rho_2)]C(\rho_2 - \rho_1)\exp\left[i\frac{k}{z}x_0(\rho_1 - \rho_2)\right]d\rho_1 d\rho_2,$$

$$\alpha_n^1\exp(i\beta_n^1) = \int\limits_{-\infty}^{\infty}\omega_n(\rho)\exp[i\varphi(\rho)]C(\rho - \rho_n^0)d\rho,$$

$$\exp(i\beta_n^0) = \exp\left[i\frac{k}{z}x_0\rho_n^0\right],$$

$$\alpha_n^2\exp\left[i\beta_n^2\right] = \iiint\limits_{-\infty}^{\infty}\omega_n(\rho_1)\omega_n(\rho_2)\omega_n(\rho_3)\exp[i\varphi(\rho_1) - i\varphi(\rho_2) - i\varphi(\rho_3)]C(\rho_n^0 - \rho_1)C^*(\rho_3 - \rho_2) \times \exp\left[-i\frac{k}{z}x_0\rho_3\right]d\rho_1 d\rho_2 d\rho_3,$$

and

$$P\exp(i\beta_p) = (\lambda^2 Rz)^{-2}\int_\Omega E(r)\mathrm{d}r.$$

Since $\alpha_n^2 \exp(i\beta_n^2) \approx \alpha_{x_0}\alpha_n^1 \exp(i\beta_{x_0} + i\beta_n^1)$, we finally derive

$$\Delta S_{18} \sim 2\Delta i\sin\frac{\theta_n}{2}\sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0)\right] \times$$

$$\times \left[\alpha_{x_0}\alpha_n^1\cos\left[\frac{\beta_{x_0} + \beta_n^1}{2}\right] + \alpha_n^1 P\cos\left[\frac{\beta_p + \beta_n^1}{2}\right]\right] +$$

$$+ 2i\Delta\sin(\theta_n)\alpha_{x_0} P\sin\left[\theta_n + 2\varphi(\rho_n^0) - \beta_{x_0} - \beta_p\right]. \quad (7.27)$$

Thus, the atmospheric distortions in each of the subapertures can be compensated separately to within the constants β_{x_0} and β_p.

Let us now write for the sharpness function S_{20}

$$\Delta S_{20} \sim \frac{\Delta A_{3H}A_4 - \Delta A_4 A_{3H}}{A_{3H}^2}, \quad (7.28)$$

where

$$A_4 = \left[\int_\Omega u^2(r)|Z(r)|^4\mathrm{d}r\right]^{1/2},$$

$$A_{3H} = \int_\Omega u(r)|Z(r)|^2\mathrm{d}r,$$

$$\Delta A_{3H} = \iint_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\exp[i\varphi(\rho_1) - i\varphi(\rho_2)]C(\rho_1 - \rho_2) \times$$

$$\times \{\exp[i\theta(\rho_1) - i\theta(\rho_2)] - 1\}\mathrm{d}\rho_1\mathrm{d}\rho_2,$$

$$\Delta A_4 = \iiiint_{-\infty}^{\infty}\omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_4)\exp[i\varphi(\rho_1) - i\varphi(\rho_2) +$$

$$+ i\varphi(\rho_3) - i\varphi(\rho_4)]B(\rho_1 - \rho_2 + \rho_3 - \rho_4) \times$$

$$\times [\exp[i\theta(\rho_1) - i\theta(\rho_2) + i\theta(\rho_3) - i\theta(\rho_4)] - 1] \times$$

$$\times \mathrm{d}\rho_1\mathrm{d}\rho_2\mathrm{d}\rho_3\mathrm{d}\rho_4,$$

$$C(\rho_1 - \rho_2) = \int_\Omega u(r)G(r - \rho_1)G^*(r - \rho_2)\mathrm{d}r,$$

and

$$B(\rho_1 - \rho_2 + \rho_3 - \rho_4) = \int_\Omega u^2(r)G(r - \rho_1)G^*(r - \rho_2) \times$$

$$\times G(r - \rho_3)G^*(r - \rho_4)\mathrm{d}r.$$

After the transformations similar to those performed for the above sharpness functions we derive

$$\Delta S_{20} \sim 2i\Delta\sin\frac{\theta_n}{2}\left[\frac{A_4}{A_{3H}^2}\alpha_n^1 - \frac{1}{A_4 A_{3H}}\alpha_n^4\right] \times$$

$$\times \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^1\right] - 2i\Delta\sin\frac{\theta_n}{2}\frac{1}{A_4 A_{3H}} \times$$

$$\times \left[2\alpha_n^2\cos\frac{\theta_n}{2}\sin\left[\theta_n + 2\varphi(\rho_n^0) - \beta_n^2\right] +\right.$$

$$\left. + \alpha_n^3\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \beta_n^3\right]\right], \quad (7.29)$$

where

$$\alpha_n^1\exp(i\beta_n^1) = \int_{-\infty}^{\infty}\omega_n(\rho)C(\rho - \rho_n^0)\exp(-i\varphi(\rho))\mathrm{d}\rho,$$

$$\alpha_n^2\exp\left[i\beta_n^2\right] = \iiint_{-\infty}^{\infty}\omega_n(\rho_1)\omega(\rho_2)\omega(\rho_3)\exp[-i\varphi(\rho_1) -$$

$$- i\varphi(\rho_2) + i\varphi(\rho_3)]B(\rho_1 - \rho_2 + \rho_3 - \rho_n^0)\mathrm{d}\rho_1\mathrm{d}\rho_2\mathrm{d}\rho_3,$$

$$\alpha_n^3\exp\left[i\beta_n^3\right] = \iint_{-\infty}^{\infty}\omega_n(\rho_1)\omega_n(\rho_2)\exp[-i\varphi(\rho_1) - i\varphi(\rho_2)]\times$$

$$\times B(2\rho_n^0 - \rho_1 - \rho_2)\mathrm{d}\rho_1\mathrm{d}\rho_2,$$

and

$$\alpha_n^4\exp\left[i\beta_n^4\right] = \int_{-\infty}^{\infty}\omega_n(\rho)\exp[-i\varphi(\rho)]B(\rho - \rho_n^0)\mathrm{d}\rho$$

and the condition $\beta_n^1 \approx \beta_n^4 - \beta_n$ is used because of the smoothness of the function $u(r)$. If β_n, β_n^2, and β_n^3 depend only on ρ_n^0, that is, on each concrete subaperture Δ_n and enter in Eq. (7.29) with the same sign, we cannot successively compensate for the distortions.

Let us now consider the last sharpness function S_{27}:

$$\Delta S_{27} \sim \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2} \cdot \frac{\Delta I(x) I_z(x) - \Delta I_z(x) I(x)}{I^2(x)} dx,$$

where

$$\Delta I(x) = \iint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\exp[i\varphi(\rho_1) - i\varphi(\rho_2)] \times$$

$$\times \varepsilon_1(\rho_1)\varepsilon_1^*(\rho_2)\exp\left[i\frac{k}{z}x(\rho_1 - \rho_2)\right] \times$$

$$\times \{\exp[i\theta(\rho_1) - i\theta(\rho_2)] - 1\} d\rho_1 d\rho_2,$$

and

$$\Delta I_z(x) = \iiiint\limits_{-\infty}^{\infty} \omega(\rho_1)\omega(\rho_2)\omega(\rho_3)\omega(\rho_4)\varepsilon^1(\rho_1)\varepsilon^{*1}(\rho_2) \times$$

$$\times \exp[i\varphi(\rho_1) - i\varphi(\rho_2) + i\varphi(\rho_3) - i\varphi(\rho_4)]\varepsilon_1(\rho_1 + \rho_2) \times$$

$$\times \varepsilon_1^*(\rho_2 + \rho_4)\{\exp[i\theta(\rho_1) - i\theta(\rho_2) + i\theta(\rho_3) -$$

$$- i\theta(\rho_4)] - 1\} d\rho_1 d\rho_2 d\rho_3 d\rho_4.$$

Simple but long transformations yield

$$\Delta S_{27} \sim 2i\Delta\sin\frac{\theta_n}{2} \left\{ \left|\varepsilon^1(\rho_n^0)\right|^2 \left|\varepsilon(2\rho_n^0)\right| \left|F_4(\rho_n^0)\right| \times \right.$$

$$\times \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \arg\varepsilon(\rho_n^0) - \arg F_4(\rho_n^0)\right] +$$

$$+ \left|\varepsilon^1(\rho_n^0)\right| \left|\varepsilon(2\rho_n^0)\right| \left|F_5(\rho_n^0)\right| \cdot \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \right.$$

$$\left. - \arg\varepsilon(\rho_n^0) - \arg\varepsilon(2\rho_n^0) - \arg F_5(\rho_n^0)\right] + \left|\varepsilon(\rho_n^0)\right| \times$$

$$\times \left|F_1(\rho_n^0)\right| \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \arg\varepsilon(\rho_n^0) - \right.$$

$$\left. - \arg F_1(\rho_n^0)\right] + \left|\varepsilon^1(\rho_n^0)\right| \left|F_2(\rho_n^0)\right| \times$$

$$\times \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \arg\varepsilon^1(\rho_n^0) - \arg F_2(\rho_n^0)\right] +$$

$$+ \left|F_3(\rho_n^0)\right| \sin\left[\frac{\theta_n}{2} + \varphi(\rho_n^0) - \arg F_3(\rho_n^0)\right] +$$

$$+ \left|\varepsilon^1(\rho_n^0)\right| \left|\varepsilon(2\rho_n^0)\right| \left|F_0(\rho_n^0)\right| \times \sin\left[\theta_n + 2\varphi(\rho_n^0) - \right.$$

$$\left.\left. - \arg\varepsilon^1(\rho_n^0) - \arg\varepsilon(2\rho_n^0) - \arg F_0(\rho_n^0)\right]\right\}, \quad (7.31)$$

where

$$F_0(\rho_n^0) = \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2} \cdot \frac{E_0(x)}{I(x)} \exp\left[-2i\frac{k}{z}x\rho_n^0\right] dx,$$

$$F_1(\rho_n^0) = \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2} \cdot \frac{I_z(x)E(x)}{I^2(x)} \exp\left[-i\frac{k}{z}\rho_n^0 x\right] dx,$$

$$F_2(\rho_n^0) = \int\limits_{\Omega_p} \frac{E_0^*(x)}{|Z^1(x)|^2} \cdot \frac{E_n^0(x)}{I(x)} \exp\left[-i\frac{k}{z}x\rho_n^0\right] dx,$$

$$F_3(\rho_n^0) = \int\limits_{\Omega_p} \frac{E_0(x)}{|Z^1(x)|^2} \cdot \frac{E_n^1(x)}{I(x)} \exp\left[i\frac{k}{z}x\rho_n^0\right] dx,$$

$$F_4(\rho_n^0) = \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2} \cdot \frac{E_n^0(x)}{I(x)} \exp\left[-i\frac{k}{z}x\rho_n^0\right] dx,$$

$$F_5(\rho_n^0) = \int\limits_{\Omega_p} \frac{1}{|Z^1(x)|^2} \cdot \frac{E_n^1(x)}{I(x)} \exp\left[-i\frac{k}{z}\rho_n^0 x\right] dx,$$

$$E_0(x) = \iint\limits_{-\infty}^{\infty} \omega_n(\rho_1)\omega_n(\rho_2)\exp[-i\varphi(\rho_1) - i\varphi(\rho_2)] \times$$

$$\times \varepsilon_1^{1*}(\rho_1)\varepsilon^*(\rho_1 + \rho_2)\exp\left[-ik\frac{x}{z}(\rho_1 + \rho_2)\right] d\rho_1 d\rho_2,$$

$$E_n^1(x) = \int\limits_{-\infty}^{\infty} \omega_n(\rho)\exp[i\varphi(\rho)]\varepsilon^1(\rho)\varepsilon(\rho + \rho_n^0) \times$$

$$\times \exp\left[ik\frac{x}{z}\rho\right] d\rho,$$

and

$$E_n^0(x) = \int\limits_{-\infty}^{\infty} \omega_n(\rho)\exp[i\varphi(\rho)]\varepsilon(\rho_n^0 + \rho) d\rho.$$

Since all the quantities $F_1(\rho_n^0)$ that enter in Eq. (7.31), are different, seeking the absolute maximum of S_{27} by the given method is apparently impossible.

3) In practice, the third way of finding the absolute maximum of the sharpness function is employed. After the

optimal position has been found, the subaperture Δ_n remains in it and does not return to the initial position. For the sharpness function S_1 this leads to replacing the coefficients β_n by β'_n, which are determined from the system

$$\alpha'_1 \exp(i\beta'_1) = \alpha_1 \exp(i\beta_1),$$

$$\alpha'_2 \exp(i\beta'_2) = \alpha_2 \exp(i\beta_2) + \Delta\{\exp[i\beta'_1 - i\varphi(\rho_1^0)] - 1\} C(\rho_1^0 - \rho_2^0),$$

and

$$\alpha'_3 \exp(i\beta'_3) = \alpha_3 \exp(i\beta_3) + \Delta\{\exp[i\beta'_2 - i\varphi(\rho_1^0)] - 1\} \times C(\rho_1^0 - \rho_2^0) + \Delta\{\exp[i\beta'_2 - i\varphi(\rho_2^0)] - 1\} C(\rho_2^0 - \quad (7.32)$$

It can be seen that in the general case the coefficients β'_n for different n are not identical. Such values θ_n, for which $\varphi(\rho_n^0) + \theta_n - \beta'_n = 0$ for any n, satisfy the condition of maximization of the sharpness function S_1. This also does not provide for the image restoration. It is apparent that all the rest of the sharpness functions can be considered in a similar way that gives the same results as in the approach 1).

It should be noted that we have made use of the simplest form of the response function, which is rectangular. However, it is of low probability that the considered methods could provide for a qualitatively different results for more complicated response functions.

In addition to the considered methods, the so-called gradient methods exist, which are applicable to operation with adaptive elements with nonlocal response (the condition (7.1) is not satisfied).[45] These methods are based on the series expansion of the phase profile $\theta(\rho)$ in terms of the response functions of the adaptive element

$$\theta(\rho) = \sum_{1=1}^{N} a_1 S_1(\rho),$$

where $S_1(\rho)$ is the response function and a_i are the controlling coefficients.

The approach for seeking the maximum is the iteration method. Starting from the zero–order approximation $a_1^{(0)}$ which is assigned based on various considerations[51] (e.g., that the profile $\theta(\rho)$ is a plane or a sphere), the values $\theta^{(0)}(\rho)$ are calculated and the sharpness function $S_1^{(0)}$ is then determined. The coefficient $a_1^{(0)}$ is then given a small increment h and

$$S'^{(0)}_h : S^{(0)}(a^{(0)} + h) - S^{(0)}(a^{(0)}) = hS'_h(a^{(0)})$$

is calculated. The subsequent approximation $a^{(1)}$ is then found and so on via the formula

$$a^{(n+1)} = a^{(n)} + (a^n) S'^{(n)}_h (a^{(n)}). \quad (7.34)$$

The considered methods of seeking similar to those suggested in Ref. 29 cannot guarantee that the sharpness function converges to maximum for arbitrary phase distributions $\varphi(\rho)$. Moreover, they do not solve the problems of secondary maxima. When there are secondary maxima, the number of the sorted states $a_1^{(j)}$ is the same as for sorting the values θ_n, what, as it has been indicated above, is unreal from the viewpoint of the speed of response and energetics.

Thus, when the existing methods of seeking the absolute maxima are used, we may only deal with the sharpness functions S_{12}, S_{14}, S_{15}, and S_{18}. It should be noted that S_{14} is of "exotic" character since it requires the comprehensive *a priori* information about the object, and S_{15} assumes an iteration algorithm in which the larger the amount of an *a priori* information about the object is the higher is the probability of convergence of this algorithm.

8. CONCLUSION

The studies of the existing sharpness functions with respect to the absolute maximum allow us to draw the following conclusions. The atmospheric phase distortions cannot be separated from the phase of the object field, and the images cannot be restored when the objects with the unknown shapes are coherently illuminated in a traditional way. This is attributed to the fact that the object field in the receiving plane and the phase distortions during time in which the atmosphere can be considered frozen do not change and are identically described mathematically. The image restoration in this case requires an *a priori* information about the object shape. The more is the amount of this information, the better (the sharpness functions S_{14} and S_{15}). The same also pertains to the phase distortion measurements by finding their optimal estimates (5.10) and (5.14).

When the objects are preliminary illuminated with linearly polarized light the image of a convex object can be constructed using the sharpness function S_9. If, however, the surface of such an object is rough and roughness is greater than λ, the speckle structure of the measured intensity should be smoothed out because of its different character for mutually orthogonal components of polarization. This consequently complicates the operation.

For untraditional illumination (formation of an artificial "beacon") the imaging can be performed, when the sharpness function is maximized by focusing into the "point" (S_{18}, S_{20}) or when it is used to form the known distribution of the illumination on the object S_{13} and S_{27}.

The object field and the phase distortions are different functions of time when the illumination is 'incoherent. Therefore, when there is no *a priori* information about the shape of the extended and incoherently illuminated objects, the atmospheric phase distortions can be compensated by maximizing the sharpness functions S_1 and S_{12} (S_2 is similar to S_1). When an *a priori* information is available, the atmospheric phase distortions are compensated by maximizing the sharpness functions S_4 and S_{10} (the information is comprehensive) or S_5 and S_{11} (the information is not comprehensive). In many cases the statistical processing of signals reflected from the coherently illuminated objects with rough surfaces reveals the analogy with the incoherent illumination. This concerns the operation with the sharpness functions S_9, S_{18}, and S_{20}, which make it possible to restore the image.

Despite the considerable amount of sharpness functions which provide for the compensation for atmospheric distortions and the restoration of images (there are thirteen of them), when they achieve the absolute maximum, the maximization algorithm implementation is a rather complicated problem. As it was indicated above, this is due to the secondary extrema which for the unknown object do not provide for discrimination between them and the absolute maximum. The available methods for seeking the maximum allow one to deal only with four (or three) sharpness functions S_{12}, S_{14}, S_{15}, and S_{18}. As for S_{14} (and in part S_{15}), it requires a comprehensive *a priori* information about the object, what is unreal in the majority of cases. In the other cases, the adaptation seems to be questionable. The sharpness function S_{18} requires for its realization that the outgoing wave be focused into a bright "point" on the object what grows into a complicated problem by virtue of the restrictions imposed by the wave divergence, the transmitting aperture, and the distance to the object. Using the sharpness function S_{12}, it is possible to find the absolute maximum for incoherent illumination of the object by compensating for the atmospheric distortions separately for each subaperture. However, there exist their own difficulties in this case. Since the frequency $|f_0|$ for S_{12} lies in the low frequency range, instantaneous atmosphere–lens OTF of $H(f_0)$ differs insignificantly from the aberration–free OTF of $H_0(f_0)$ (see, e.g., Refs. 2, 5, and 6). Therefore, when one takes into account the additive noise being always presented both in the image plane and in the Fourier plane the sensitivity S_{12} to adaptation for each of the subapertures is low.

Nevertheless, despite the above indicated difficulties, S_{12} is the only sharpness function that provides for restoration of the image of the unknown extended object for incoherent illumination. As for coherent illumination (without focusing into the "point"), the problem has not been yet solved, since it is not clear, in what way the absolute maximum of the sharpness functions S_9, S_{13}, and S_{27} that provides for the image restoration, can be found.

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