SOME ASPECTS OF THE THEORY OF ASTRONOMICAL REFRACTION IN THE EARTH'S ATMOSPHERE

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New technique of theoretical treatment of astronomical refraction in three–dimensional inhomogeneous atmosphere is discussed. This technique is based on integral representation of the ray equation, which reduces the refraction problem to the solution of algebraic equations for refraction angles and length of any part of the ray trajectory bent by the Earth's atmosphere. Using model of spherically layered atmosphere as an example it is demonstrated that the proposed technique can yield a prescribed accuracy of determination of the refraction within the range of apparent zenith angles $0° < z < 90°$.

1. INTRODUCTION

Refraction of electromagnetic waves in the inhomogeneous Earth's atmosphere is one of the main factors restricting the accuracy of observation methods in classical and modern astrometry. Despite the long history of studying the astronomic refraction,[1–3] there are still some problems in the theory of this phenomenon. In particular, there is no comprehensive theoretical investigation of anomalies in the astronomic refraction for the use of three–dimensional inhomogeneous Earth's atmosphere, the questions of refraction in the near–horizon zone[4] are poorly developed.

Very often the consideration of these questions uses *a priori* analytical models of the three–dimensional atmospheric profile enabling exact integration of refraction equations,[5] or numerical solution of the initial beam equations, for instance, by Gartser's scheme.[6] However, the accuracy of integrated models can be insufficient, and numerical solution of the beam equations requires preliminary determination of the three–dimensional profile of air refractivity what is not always feasible in practice.

In this connection, we propose in this paper a new approach to solving the refraction problems. The aim of this approach is to overcome the restrictions of the known methods and fill up the gaps in the theory of astronomic refraction in the three–dimensional inhomogeneous Earth's atmosphere.

2. GENERAL RELATIONS

Within the framework of the proposed method, we use the beam equation of the geometric optics[7]

$$\frac{\mathrm{d}}{\mathrm{d}\,\sigma}(n\,\mathbf{l}) = \nabla n\,, \tag{1}$$

where ∇ is the gradient operator, $n = n(\mathbf{r})$ is the refractive index of air, $\mathbf{l} = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}\sigma}$ is the unit vector tangent to the beam trajectory, σ is the beam coordinate, and $\mathbf{r}$ is the radius vector of a point in the beam.

Consider the first integral of Eq. (1):

$$n_L\,\mathbf{l}_L - n_0\,\mathbf{l}_0 = \int\limits_0^D \nabla n\,\mathrm{d}\sigma\,, \tag{2}$$

where n_0 and $\mathbf{l}_0$ are the refractive index of air and the unit vector tangent to the beam trajectory at the observation point, respectively; n_L and $\mathbf{l}_L$ are analogous parameters at the point of intersection of the beam trajectory with the upper boundary of the Earth's atmosphere (we assume that the beam intersecting this upper boundary surface of the Earth's atmosphere propagates then toward the object observed already in vacuum along a straight line, the direction of which is given by the vector $\mathbf{l}_L$); D is the length of the beam trajectory section bent in the Earth's atmosphere, i.e., the length of the beam trajectory from the initial observation point to the point of its intersection with the upper boundary of the atmosphere.

By Euler–Maclaurin expansion[8] Eq. (2) can be presented in the form[9]

$$n_L\,\mathbf{l}_L - n_0\,\mathbf{l}_0 = \frac{\nabla\,n_0 + \nabla\,n_L}{2}\,D - \frac{D^2}{12}(\nabla\,n_L^{\mathrm{I}} - \nabla\,n_0^{\mathrm{I}}) + $$
$$+ \frac{D^4}{720}(\nabla\,n_L^{\mathrm{III}} - \nabla\,n_0^{\mathrm{III}}) - \frac{D^6}{30240}(\nabla\,n_L^{\mathrm{V}} - \nabla\,n_0^{\mathrm{V}}) + \dots\,, \tag{3}$$

where $\nabla n^{\mathrm{I}} = \dfrac{\mathrm{d}\nabla n}{\mathrm{d}\sigma}$, $\nabla n^{\mathrm{III}} = \dfrac{\mathrm{d}^3\nabla n}{\mathrm{d}\sigma^3}$, $\nabla n^{\mathrm{V}} = \dfrac{\mathrm{d}^5\nabla \mathrm{n}}{\mathrm{d}\sigma^5}$, ... ; $\dfrac{\mathrm{d}}{\mathrm{d}\sigma}$ is the differentiation operator with respect to the beam coordinate; the indices "0" and "L" denote here, as in the above, the observation point and the point of intersection of the beam and the upper boundary of the atmosphere, respectively.

Since $n_L = 1$ and $\nabla n_L = \nabla n_L^{\mathrm{I}} = \nabla n_L^{\mathrm{III}} = \dots = 0$, by definition of the upper boundary of the atmosphere, the initial equation of the astronomic refraction (3) can be reduced to a simpler form

$$\mathbf{l}_L - n_0\,\mathbf{l}_0 = \nabla n_0\frac{D}{2} + \nabla\,n_0^{\mathrm{I}}\frac{D^2}{12} - \nabla\,n_0^{\mathrm{III}}\frac{D^4}{720} + \nabla\,n_0^{\mathrm{V}}\frac{D^6}{30240} - \dots\,. \tag{4}$$

The vector equation (4) corresponds to a system of three scalar equations for three unknown values α, α_T, and D (it is easy to take into consideration the angles α and α_T of vertical and lateral refraction when writing vector equation (4) in terms of projections accounting for the fact that the vector $\mathbf{l}_L$ describes the true direction to the object observed while the vector $\mathbf{l}_0$ – the apparent one; this procedure is described in Ref. 10 for the Cartesian coordinate system).

One should emphasize two important features of Eq. (4) as an integral representation of the beam equation (1). First, all the coefficients in Eq. (4) are determined near the observation point; therefore, in such an approach, it is not necessary to find the vertical profile of the refractive index of the air in order to calculate the refraction. Second, there are no restricting assumptions concerning the profile of the refractive index when deriving Eq. (4), Eq. (4) is valid under conditions quite common for geometric optics approximation,[7] so that the profile can be sufficiently arbitrary.

The applicability limits of Eq. (4) depend on the number of expansion terms taken into account in its right–hand side. The simplest situation, when only the first term (proportional to D^1) of the series in the right–hand side of Eq. (4) is taken into account, is considered in Ref. 10 where the generalizations of Oriani–Laplace theorem[1,11] are obtained for the case of three–dimensional inhomogeneous atmosphere, i.e., horizontal and lateral refraction anomalies connected with horizontal components of the air refractivity gradient are determined. The results from Ref. 10 well agree with those in Ref. 4 and they are valid for zenith angles $z < 70°$. A wider range of zenith angles is covered in Ref. 12 where the solution of Eq. (4) was obtained with regard to terms proportional to D^1, D^2, and D^4.

The aim of the present paper is to consider the case of arbitrary zenith angles and to obtain the solution taking into account an arbitrary number of expansion terms in the right–hand side of Eq. (4).

3. REFRACTION IN SPHERICALLY LAYERED MODEL OF THE ATMOSPHERE

The necessity of taking into account additional terms of the series (4) arises for exact calculations of the refraction angle at zenith distances $z > 70°$. The question about its convergence becomes a subject of special importance. We consider this question for a spherically layered atmosphere when the lateral diffraction is absent and the problem is simplified (only the vertical refraction angle α and the length D remain unknown).

In this case $n = n(r)$, $\nabla n = g\mathbf{r}/r$, where $\mathbf{r}$ is the radius vector of a point in spherical coordinate system with the origin at the center of the Earth, $r = |\mathbf{r}|$ is the distance from the center of the Earth to the point with the radius vector $\mathbf{r}$, and $g = \dfrac{\mathrm{d}n}{\mathrm{d}r}$ is the vertical gradient of air refractivity.

Taking into account the fact that in this case the beam is a plane curve, the derivative of each order of the value ∇n with respect to the beam coordinate can be presented in the form

$$\nabla n^{(i)} = \frac{\mathrm{d}^i\, \nabla n}{\mathrm{d}\, \sigma^i} = \frac{\mathbf{r}_0}{r_0} R_i + \mathbf{l}_0\, L_i\,, \qquad (5)$$

where $(i) = 0$, I, II, III, IV, V, ... when $i = 0, 1, 2, 3, 4, 5, \ldots$, and the coefficients R_i and L_i are defined directly, as a result of calculation of the corresponding derivatives of the value ∇n. For instance, the obvious formula

$$\nabla\, n_0^0 = \nabla n_0 = g\mathbf{r}_0 \,/\, r_0$$

implies that

$$R_0 = g\,, \quad L_0 = 0\,.$$

Furthermore, calculating the derivative

$$\nabla\, n_0^{\mathrm{I}} = \frac{\mathrm{d}\, \nabla\, n_0}{\mathrm{d}\, \sigma} = \frac{\mathbf{r}_0}{r_0} \cdot \frac{(\mathbf{l}_0\, \mathbf{r}_0)}{r_0} \left[g' - \frac{g}{r_0} \right] + \mathbf{l}_0\, \frac{g}{r_0}\,,$$

where $g' = \dfrac{\mathrm{d}g}{\mathrm{d}r}$, we see that

$$R_1 = \frac{(\mathbf{l}_0 \mathbf{r}_0)}{r_0} \left[g' - \frac{g}{r_0} \right]; \quad L_1 = \frac{g}{r_0}\,.$$

By making similar calculations for $\nabla n_0^{\mathrm{III}}$ we find

$$R_3 = \frac{(\mathbf{l}_0 \mathbf{r}_0)}{r_0} \left\{ 3\, \frac{g''}{r_0} - 9\, \frac{g'}{r_0^2} + 9\, \frac{g}{r_0^3} + 3\, \frac{g\, g''}{n_0} + \frac{(g')^2}{n_0} - \right.$$

$$- 7\, \frac{g\, g'}{n_0 r_0} - 3\, \frac{g'\, g^2}{n_0^2} + 6\, \frac{g^2}{n_0\, r_0^2} + \frac{g^3}{n_0^2 r_0} +$$

$$+ \frac{(\mathbf{l}_0\, \mathbf{r}_0)^2}{r_0^2} \left[\mathrm{g}''' - \frac{6\, g''}{r_0} + 15\, \frac{g'}{r_0^2} - 15\, \frac{g}{r_0^3} - \frac{(g')^2}{n_0} - \right.$$

$$\left.\left. - 3\, \frac{g\, g''}{n_0} + 3\, \frac{g^2\, g'}{n_0^2} + 11\, \frac{g\, g'}{n_0 r_0} - 3\, \frac{g^3}{n_0^2 r_0} - 10\, \frac{g^2}{n_0 r_0^2} \right] \right\},$$

where

$$g'' = \frac{\mathrm{d}^2\, g}{\mathrm{d}\, r^2}\,, \quad g''' = \frac{\mathrm{d}^3\, g}{\mathrm{d}\, r^3}\,,$$

$$L_3 = 3\, \frac{g'}{r_0^2} - 3\, \frac{g}{r_0^3} + 3\, \frac{g\, g'}{n_0 r_0} - 3 \frac{g^2}{n_0 r_0^2} - \frac{g^3}{n_0^2 r_0} +$$

$$+ \frac{(\mathbf{l}_0\, \mathbf{r}_0)^2}{r_0^2} \left[-7\, \frac{g\, g'}{n_0\, r_0} + 7 \frac{g^2}{n_0 r_0^2} + 3\, \frac{g^3}{n_0^2 r_0} + 3\, \frac{g''}{r_0} - 9\, \frac{g'}{r_0^2} + 9\, \frac{g}{r_0^3} \right],$$

and so on.

Then, multiplying Eq. (4) in turn by $\mathbf{l}_0$ and $\dfrac{\mathbf{r}_0}{r_0}$, we obtain, taking into account Eq. (5), the system

$$\cos\alpha - n_0 = A\,, \qquad \cos(\alpha + z) - n_0 \cos z = B\,, \qquad (6)$$

where $\alpha = \arccos(\mathbf{l}_0\, \mathbf{l}_L)$ is the angle of vertical refraction; $z = \arccos(\mathbf{l}_0 \dfrac{\mathbf{r}_0}{r_0})$ is the apparent zenith distance (zenith angle); the values A and B are defined by the relations

$$A = \frac{D}{2}\, R_0 \cos z + \frac{D^2}{12}\, (R_1 \cos z + L_1) - \frac{D^4}{720}\, (R_3 \cos z + L_3) +$$

$$+ \frac{D^6}{30240}\, (R_5 \cos z + L_5) - \ldots$$

$$B = \frac{D}{2}\, R_0 + \frac{D^2}{12}\, (R_1 + L_1 \cos z) - \frac{D^4}{720} (R_3 + L_3 \cos z) +$$

$$+ \frac{D^6}{30240}\, (R_5 + L_5 \cos z) - \ldots\,.$$

The system of equations (6) allows one to find the unknown values α and D. By introducing the designations

$$T = \frac{D^2}{12} L_1 - \frac{D^4}{720} L_3 + \frac{D^6}{30240} L_5 - \dots , \quad (7)$$

$$S = \frac{D}{2} R_0 + \frac{D^2}{12} R_1 - \frac{D^4}{720} R_3 + \frac{D^6}{30240} R_5 - \dots , \quad (8)$$

one can conveniently present the solution of this system in the form

$$\alpha = (-1)^k \arcsin \{\sin z [n_0 + T]\} + k\pi - z , \quad (9)$$

where $k = 0$ when $0 \le \alpha + z \le \pi/2$, $k = 1$ when $\alpha + z > \pi/2$, and the value D (the length of a portion of the beam trajectory bent in the Earth's atmosphere) determining the parameter T can be found by solving the algebraic equation

$$1 = [(n_0 + T) \cos z + S]^2 + \sin^2 z [n_0 + T]^2 , \quad (10)$$

where T and S are given by formulas (7) and (8), respectively.

The obtained solution generalizes the particular case of the system (6) solution considered in Ref. 12 that takes into account only the expansion terms proportional to D, D^2, and D^4 to the general case of arbitrary number of the expansion terms.

Note that one can assume $T = 0$ when $r_0 \to \infty$ ("flat Earth" approximation); then Eqs. (9) and (10) become independent and, by expanding Eq. (9) in a power series ($n_0 - 1 \ll 1$), we obtain the well-known expression from the Oriani–Laplace theorem[1,2]

$$\alpha = (n_0 - 1) \tan z , \quad (11)$$

which is valid according to many numerical calculations, at $z < 70°$ with the error not exceeding some tenths of a seconds of an arc.[11]

For $z = 90°$ the formulas (9) and (10) give the relation

$$\alpha = |g|\, D/2 ,$$

formally coinciding with the well-known[13] relationship used for estimating the geodesic refraction angle along a horizontal path of the length D.

Within the framework of the theory stated in the present paper it is easy to estimate analytically the applicability criterion of Oriani–Laplace theorem (11). Using the above-mentioned formulas (6)–(10) this estimation can be presented in the form

$$\tan^2 z \ll |g'|\, r_0 / |g| . \quad (12)$$

For the angles $z > 70°$ when the condition (12) is not satisfied, $T \ne 0$, and one should solve Eq. (10) in order to calculate the refraction α. The number of expansion terms to be taken into account in the relations (7) and (8) depends on the accuracy of the refraction angle calculations and the convergence rate of the series (7) and (8).

Let us consider the question of convergence in the most unfavorable case when $z = 90°$ (note that the canonical expansion of the classical refraction integral diverges when $z = 90°$ (Ref. 1)). In the given case $R_0 = g$, $R_1 = R_3 = \dots = 0$, and the system of equations (6) has the form

$$\cos\alpha - n_0 = \frac{D^2}{12} L_1 - \frac{D^4}{720} L_3 + \frac{D^6}{30240} L_5 - \dots , \quad (13)$$

$$-\sin\alpha = g \frac{D}{2} . \quad (14)$$

By determining D from Eq. (14) and substituting it into Eq. (13), after expanding cos α and sin α in power series (α ≪ 1), we obtain

$$0 = (n_0 - 1) + \alpha^2 B_2 + \alpha^4 B_4 + \alpha^6 B_6 + \dots , \quad (15)$$

where

$$B_2 = \frac{1}{2} + \frac{L_1}{3|g|^2} ; \quad B_4 = -\frac{1}{24} - \frac{L_1}{9|g|^2} - \frac{L_3}{45|g|^4} ;$$

$$B_6 = \frac{1}{720} + \frac{2}{135}\left[\frac{L_1}{|g|^2} + \frac{L_3}{|g|^4}\right] + \frac{2}{945}\frac{L_5}{|g|^6} ; \quad |g| = -g .$$

Approximate estimations of the coefficients in Eq. (15) made for standard (polytropic) atmospheric model and $\lambda = 0.59$ μm, $r_0 = 6367.5$ km give the following values of the parameters: $B_2 \simeq -1.51$, $B_4 \simeq -0.55 \cdot 10^4$, $B_6 \simeq -0.22 \cdot 10^8$, and $(n_0 - 1) \simeq 2.77 \cdot 10^{-4}$. Taking into account that $\alpha \simeq 2000''$, i.e., $\alpha \simeq 10^{-2}$ (Ref. 1) when $z = 90°$, and substituting the values of α, $(n_0 - 1)$, B_2, B_4, and B_6 into Eq. (15), we obtain a convergent numerical series

$$0 \simeq 2.77 \cdot 10^{-4} - 1.51 \cdot 10^{-4} - 0.55 \cdot 10^{-4} - 0.22 \cdot 10^{-4} - \dots . \quad (16)$$

Thus, from Eq. (16) one can immediately see that even in the most unfavorable situation with respect to the convergence conditions (when $z = 90°$) the series in the right-hand side of Eq. (15), which describes the refraction angle α, converges.

The convergence of the obtained expansions is illustrated in Fig. 1 in which the dependences of the refraction angle α on the zenith distance z are presented. The curve *1* gives here the exact values of α obtained by numerical integration; the curve *2* shows the results of calculations by formulas (7)–(10) with regard to terms proportional to D, D^2, and D^4; the curve *3* is obtained with regard to terms proportional to D, D^2, D^4, and D^6; the curve *4* is calculated by the formula (11). All the calculations have been performed using the aforesaid numerical values of the parameters used in estimation of the coefficients in Eq. (15). One can see that the dependence (9) obtained in the present paper for the refraction angle tends to the exact solution with the increase of expansion terms taken into account in Eqs. (7) and (8).

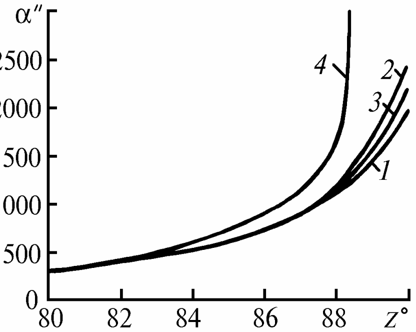

FIG. 1. The astronomic refraction angle α vs. the zenith distance z: exact calculation (1), calculation by formulas (7)–(10) with regard to terms proportional to D, D^2, and D^4 (2), similar calculation with regard to terms proportional to D, D^2, D^4, and D^6 (3), and calculation by formula (11) (curve 4).

From Fig. 1 one can also see that the problems on convergence become less severe when $z < 87 \ldots 89°$. The calculations performed made it possible to specify the preliminary data[12] and to conclude that it is sufficient to take into account only the terms with powers of D not more than 4 in Eqs. (7)–(10) in order to determine the refraction angle with the error no more than 0.5" in the range of zenith angles $0° \le z \le 80°$ (exceeding by 10° the range in which the Oriani–Laplace theorem is valid).

The calculations also show that the accuracy of the formulas (7)–(10) at $z < 70°$ exceeds the accuracy of Oriani–Laplace theorem almost by two orders even with regard to only the terms proportional to D, D^2, and D^4.

4. CONCLUSION

Thus, in the present paper a new approach to the theoretical solution of the problem on determining astronomic refraction in the three–dimensionally inhomogeneous Earth's atmosphere is developed. It is shown that within the framework of the given approach one can provide the given accuracy of the calculation of the vertical refraction angle in the whole range of apparent zenith angles $0° \le z \le 90°$ for the model of spherically layered atmosphere using only meteorological data obtained near the observation point.

Further development of the discussed approach can be connected with a more careful investigation of the possibilities of the model of spherically layered atmosphere (in particular, on the basis of numerical experiments using actual profiles from radiosounding of the atmosphere) and with the consideration of the general case of astronomic refraction in near–horizon zone for the three–dimensional inhomogeneous Earth's atmosphere. It is convenient to use Eq. (4) as an initial equation enabling the investigation of horizontal and lateral refraction anomalies in this general case.

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