ESTIMATION OF COORDINATES OF A SOURCE FROM THE SIGNALS RECORDED AT FIVE REFERENCE POINTS

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This paper deals with the problem of statistical estimation that arises in calculation of coordinates of a source from the data obtained with a satellite system. Calculation formulas involve time lags of signal arrival at reference points of the system. These time lags are evaluated from pairwise comparison between the versions (realizations) of signals recorded at reference points using the first moment of the impulse transfer characteristic of an adaptive filter which transforms one signal version into another provided that the error of transformation is at its minimum.

1.The problem of determination of the three–dimensional radius vector $\mathbf{r}$ of optical source from signals measured at five different points (called reference points) of space with the known radius vectors $\mathbf{r}_i$ ($i = 1, ..., 5$) is considered. Measurements can be carried out, for example, with the use of spacecrafts (satellite constellation) being part of satellite system of global orientation.[1]

Time lags between signal arrival at reference points τ_{j1}, $j = 2, 3, 4, 5$ and covariance matrix of errors in their determination ξ_{j1} are the initial data for calculation of the coordinates of a source

$$R = [r_{kj}],\ r_{kj} = E\{(\xi_{k1} - T)\,(\xi_{j1} - T)^*\},\ j,\ k = 2, 3, 4, 5\ .$$

Here ξ_{j1} are the Gaussian random variables with the mean ("synchronization shift") $T = E\{\xi_{j1}\}$; $\tau_{j1} = t_j - t_1$; t_j and t_1 are the instants of signal arrival at the points j and 1, respectively; ξ_{j1} ($j = 2, 3, 4, 5$) are the errors in estimating the lags between the signal "copies" recorded at different points of space, which are weakly disturbed and possibly scaled up or down. The systematic error associated with the constant component T in the random variables ξ_{j1} may be due to, for example, choice of one of these copies, say, the first, as the initial copy, with which the others are compared. The random components of errors are considered to be Gaussian because they are largely due to the intrinsic noise of a measuring device and filter (estimator), generally approximated by the Gaussian processes, rather than due to the effect of a signal propagation medium.

Calculations are carried out with the use of the equations

$$|\mathbf{r} - \mathbf{r}_j| - |\mathbf{r} - \mathbf{r}_1| = c(\tau_{j1} + \xi_{j1})\ , \qquad j = 2, \ldots, 5\ , \qquad (1)$$

following from the geometric consideration of the problem. In these equations

$$|\mathbf{r} - \mathbf{r}_j| = \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}$$

is the modulus of the vector $\mathbf{r} - \mathbf{r}_j$ or the distance from the radiating source to the jth reference point; c is the light velocity, which is taken to be equal to unity by corresponding scaling; $E\{\xi_{j1}\}$ is the mathematical expectation (mean value) of the random variable ξ. Introducing the designations

$$\upsilon = \begin{bmatrix} \tau_{21} \\ \cdot \\ \cdot \\ \cdot \\ \tau_{51} \end{bmatrix},\quad N = \begin{bmatrix} \xi_{21} - T \\ \cdot \\ \cdot \\ \cdot \\ \xi_{51} - T \end{bmatrix},$$

$$L(\theta) = \begin{bmatrix} |\mathbf{r} - \mathbf{r}_2| - |\mathbf{r} - \mathbf{r}_1| - T \\ \cdot \\ \cdot \\ \cdot \\ |\mathbf{r} - \mathbf{r}_5| - |\mathbf{r} - \mathbf{r}_1| - T \end{bmatrix}, \qquad (2)$$

we may rewrite Eq. (1) in the form

$$\upsilon = L(\theta) + N\ , \qquad (3)$$

where υ is the observed Gaussian random four–dimensional vector with the mean value $L(\theta)$ and covariance matrix R; θ is the desired vector–parameter

$$\theta = \begin{bmatrix} \mathbf{r} \\ T \end{bmatrix}. \qquad (4)$$

2. The solution of Eqs. (2) and (3) is the well–known problem of statistical estimation.[2] It is necessary to obtain in general the m–dimensional vector–parameter θ being optimal for the mean square error from the observed n–dimensional random Gaussian vector υ with the probability density depending on θ

$$p(\upsilon\,|\,\theta) = \frac{1}{(2\,\pi)^{n/2}\,|R\,|^{1/2}} \exp\left\{-\frac{1}{2}\Big(R^{-1}(\upsilon - L(\theta)),\ (\upsilon - L(\theta))\Big)\right\}, \qquad (5)$$

where $|R|$ is the determinant of the matrix R and $(\xi,\ \eta) \equiv \{\xi,\ \eta^*\}$ is the scalar product of the vectors ξ and η in the Hilbert space of random variables (asterisk denotes Hermitian conjugation). As applied to satellite radionavigation, this problem was considered in Ref. 3.

For sufficiently smooth transform $L(\theta)$, having derivatives of any order, the minimum Rao–Cramer matrix (minimum second moments of the estimation error

$$P(\theta) = E_\theta\left\{(\hat{\theta} - \theta)\,(\hat{\theta} - \theta)^*\right\}$$

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for fixed θ from the class of all possible linear estimates of $\hat{\theta}$) on account of Eq. (5) and the equality following from this equation

$$\left(\nabla_\theta \log p(\upsilon|\theta)\right)^* = G^*(\theta)\, R^{-1}\, (\upsilon - L(\theta))$$

is determined by the expression[3,4]

$$E\{\left(\nabla_\theta \log p(\upsilon|\theta)\right)^* \left(\nabla_\theta \log p(\upsilon|\theta)\right)\} = G^*(\theta)\, R^{-1}\, G(\theta)\ ,$$

where $R = E[(\upsilon - L(\theta))\,(\upsilon - L(\theta))^*]$, and $G(\theta)$ is the gradient ($n \times m$ matrix) of the vector function $L(\theta)$

$$G(\theta) = \nabla_\theta\, L(\theta)\ . \tag{6}$$

Since $L(\theta)$ is nonlinear transform, it is expedient "to localize" the problem taking a certain "nominal" value of θ_0 and estimating small perturbation of this quantity. Assuming that

$$L(\theta) = L(\theta_0) + G(\theta_0)\,(\theta - \theta_0)$$

and introducing the new variables

$$\tilde{\upsilon} = \upsilon - L(\theta)\ ,\ \ \tilde{\theta} = \theta - \theta_0\ ,$$

we linearize the problem reducing Eq. (3) to the form

$$\upsilon = G(\theta_0)\,\tilde{\theta} + N\ . \tag{7}$$

For the Gaussian vector N the estimate of the maximum likelihood (EML), which maximizes $p(\upsilon \mid \theta)$ for θ at any υ and satisfies the equation

$$\nabla_\theta \log p(\upsilon|\theta) = 0\ ,$$

is given by the formula

$$\hat{\theta} = \theta_0 + \left(G^*(\theta_0)\, R^{-1}\, G(\theta_0)\right)^{-1} G^*(\theta_0)\, R^{-1}\, (\upsilon - L(\theta_0)). \tag{8}$$

For Gaussian signals the EML coincides with the estimate by the least square technique.[5] This is an effective unbiased estimate of the vector–parameter θ. The error covariance matrix for this estimate is the minimum Rao–Cramer matrix and is equal to

$$P(\theta) = \left(G^*(\theta_0)\, R^{-1}\, G(\theta_0)\right)^{-1}\ . \tag{9}$$

If the correlation between ξ_{j1}, $j = 2, 3, 4, 5$ can be neglected and the variances $D\xi_{j1} = \sigma^2$ are equal, we will have

$$R^{-1} = I/\sigma^2\ ,$$

where I is the unit $n \times n$ matrix. Equations (8) and (9) are simplified for this case and take the form

$$\hat{\theta} = \theta_0 + \left(G^*(\theta_0)\, G(\theta_0)\right)^{-1} G^*(\theta_0)\, (\upsilon - L(\theta_0))\ , \tag{10}$$

$$P(\theta) = \left(G^*(\theta_0)\, G(\theta_0)\right)^{-1} \sigma^2\ . \tag{11}$$

The matrix $G(\theta_0)$ remains to be determined. Introducing the designations

$$\theta_0 = \begin{bmatrix} \mathbf{r}_0 \\ T \end{bmatrix},\ \ \mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},\ \ \mathbf{r}_j = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix},$$

we derive from Eqs. (6) and (2)

$$G(\theta_0) = (-1) \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 1 \\ \alpha_2 & \beta_2 & \gamma_2 & 1 \\ \alpha_3 & \beta_3 & \gamma_3 & 1 \\ \alpha_4 & \beta_4 & \gamma_4 & 1 \end{pmatrix}, \tag{12}$$

where

$$\left.\begin{aligned} \alpha_j &= (x_j - x_0)/a_j - (x_1 - x_0)/a_1\ , \\ \beta_j &= (y_j - y_0)/a_j - (y_1 - y_0)/a_1\ , \\ \gamma_j &= (z_j - z_0)/a_j - (z_1 - z_0)/a_1\ , \end{aligned}\right\} \tag{13}$$
$$a_j \equiv |\mathbf{r}_j - \mathbf{r}_0|\ ,\ \ j = 1, \ldots, 5\ .$$

Substituting $\hat{\theta}$ for θ_0, the calculation from these equations can be repeated to obtain a better estimate of θ. The iteration process can be continued till θ becomes so small that the following iterations give no significant improvement in the results.[5] The starting value of θ_0 is determined by the solution of the system of four equations

$$|\mathbf{r} - \mathbf{r}_j| - |\mathbf{r} - \mathbf{r}_1| = \tau_{j1} + T\ , \qquad j = 2, 3, 4, 5\ , \tag{14}$$

with four unknowns x, y, z, and T. Equations (14) are derived from Eq. (1) (for $c = 1$) ignoring the small quantities $\xi_{j1} - T$.

3. The values τ_{j1} and σ^2 which enter in the calculation relations for $\hat{\theta}$ and $P(\theta)$ are determined from the signals measured at the reference points. In radar technique with the known shape of recorded signal its arrival time τ is determined as the time t_{max} at which the signal envelope at the output of a square–law detector, connected to the output of a matched filter,[6] reaches its maximum. Therefore, $\tau = t_{max} - T'$, where T' is the time lag due to the filter. In the examined case of optical signal of unknown shape another technique based on a comparison of two versions (realizations) of recorded signal is more convenient. These signal versions are fed by pairs in different combinations (predelayed at a certain "nominal" time T_{j1}) into the input and output of an adaptive digital filter with tunable impulse transfer characteristic $h_i(n)$, $i = 0, 1, \ldots, N-1$ of finite duration N (see Refs. 7 and 8). The shapes of this characteristic are recorded for each pair at different time intervals of signal recording period.

The operation of the filter is described by the convolution equation

$$\hat{d}(n) = \sum_{i=0}^{N-1} h_i(n)\, x(n - i) = w_N^*(n)\, x_N(n)\ ,$$

where $x(n)$ is the signal fed into the filter input, $\hat{d}(n)$ is the filter output signal,

$w_N^*(n) = [h_0(n), \ldots, h_{N-1}(n)]$,
$x_N^*(n) = [x(n), \ldots, x(n - N + 1)]$.

The optimal filter characteristic $h_i(n)$ for minimum square error

$$\varepsilon(n) \equiv E[\,|d(n) - \hat{d}(n)|^2]$$

is determined by the system of standard equations which are written in the matrix form as follows[8]:

$$R_{NN}\, w_n = P_N ,$$

where $R_{NN} = E\{x_N(n)\, x_N^*(n)\}$ is the correlation matrix for the input signal, $P_N = E\{d(n)\, x_N(n)\}$ is the cross–correlation vector. Having determined the form of the function $h_i(n)$ versus i for the fixed n for the filter which transforms the ith version of the signal to the jth version, the τ_{ji} is calculated by the formala

$$\tau_{ji} = T_{ji} + \eta_{ji} , \qquad (15)$$

where η_{ji} is the center of gravity (the first moment) of the function $h_m(n)$, and T_{ji} is the nominal (guessed) value of the time lag. The variance σ^2 is determined by the expression

$$\sigma^2(n) = 2\,\tilde{\sigma}^2 \,/\, |x'(n)|^2 , \qquad (16)$$

where $\sigma^2(n)$ is the variance of the measurement errors, $x'(n)$ is the derivative of the signal $x(t)$ at the instant $t = nt_0$, and t_0 is the sample step.

The formulas for τ_{ji} and σ^2 estimation are easy interpreted when analyzing the filtration process in continuos time. Allowing for small difference between recorded versions of signal and closeness of the impulse transfer characteristic of the filter $h(t)$ to the Dirac delta function, we may use the approximation formula for the inversion of the operation of convolution performed by the filter[9]

$$x(t) \approx \frac{1}{m_0}\left[\hat{d}(t + \eta) - \frac{\zeta^2}{2}\hat{d}''(t + \eta) + \ldots\right] , \qquad (17)$$

where $\eta = m_1/m_0$, $\zeta^2 = m_2/m_0 - \eta^2$, $m_0 \approx 1$, m_1 and m_2 are the moments of the function $m_n = \int_{-\infty}^{\infty} t^n\, h(t)\, \mathrm{d}\,t$ of the zeroth, first, and second orders. The parameter σ^2 can be found as the variance of the random variable δ in the equation

$$f_0(t) + \varepsilon_1(t) \approx f_0(t + \delta) + \varepsilon_2(t) \approx f_0(t) + \delta\, f_0'(t) + \varepsilon_2(t) ,$$

which is derived from Eq. (17) when we ignore the small terms containing the factors ζ^2 in the first and higher powers by substituting

$$x(t) = f_0(t) + \varepsilon_1(t) ,\ \hat{d}(t) \approx d(t) = f_0(t + \delta) + \varepsilon_2(t) .$$

Here $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are the deviations of the recorded versions from the true shape of transmitted signal $f_0(t)$. They are considered to be independent random variables with zero mean and variance $D\varepsilon_1 = D\,\hat{\varepsilon}_2 = \tilde{\sigma}^2$.

4. The systematic error T is small as compared with the time lags τ_{i1} ($T \ll \tau_{i1}$ for any i). Therefore, as the initial estimate of θ by Eq. (14), the nominal value of T_0 may be set zero. Then from Eq. (14) we derive the system of three linear algebraic equations

$$a_i\, x + b_i\, y + c_i\, z = d_i , \qquad i = 3, 4, 5 , \qquad (18)$$

to calculate the three other components of the vector θ_0. The coefficients of these equations

$$a_i = x_i - x_1 - \frac{\tau_{i1}}{\tau_{21}}(x_2 - x_1) ,$$

$$b_i = y_i - y_1 - \frac{\tau_{i1}}{\tau_{21}}(y_2 - y_1) ,$$

$$c_i = z_i - z_1 - \frac{\tau_{i1}}{\tau_{21}}(z_2 - z_1) ,$$

$$d_i = \frac{1}{2}\left[\rho_i^2 - \rho_1^2 - \tau_{i1}^2 - \frac{\tau_{i1}}{\tau_{21}}(\rho_2^2 - \rho_1^2 - \tau_{21}^2)\right]$$

for the known parameters x_k, y_k, z_k, and $\rho_k^2 = x_k^2 + y_k^2 + z_k^2$ are determined from the experimentally measured values of τ_{k1}, $k = 2, 3, 4, 5$.

The practically important problem of optimal estimation of the coordinates of isotropic source of pulsed optical radiation from the data of satellite observations has been considered. The calculation relations have been given when the signal is recorded at five spaced points. The three coordinates x, y, and z of the source and the systematic error T in estimating the time lags of signal arrival at reference points have been determined. The problem is solved by the direct method of inversion of the nonsingular 4×4 data matrix with simultaneous finding of the matrix of errors in estimating the covariance P, which is the minimum Rao–Cramer matrix. In the case of data surplus, when the number of reference points is larger than 5, the direct method of estimating the parameters x, y, z, and T and the covariance matrix P can be substituted by recursion using the Calman filter, for example.

The time lags of signal arrival at different reference points is proposed to estimate with the use of the adaptive digital filter operating in the regime of identification of unknown system, when one version of the signal is fed at its input and its output is compared with the another version. Time lag can be estimated from the position of signal maximum or of the midpoint of the impulse response characteristic of the filter system. The method proposed is analogous to the well–known technique for estimation of time lags from the position of maxmum of the cross–correlation function for signals being compared,[10] but has the advantage that the peak of the corresponding curve is more sharply pronounced.

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