TWO–PARAMETRIC MODEL OF THE SCATTERING PHASE FUNCTION

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Based on the Barteneva classification of scattering phase functions, two parameters characterizing the scattering phase function have been obtained. The formulas have been derived that relate these parameters to meteorological parameters. The algorithm have been constructed that specifies the scattering phase function with two parameters to be preset. The known analytical formulas for approximation of the scattering phase functions have been tested as applied to the constructed model, and new formula for the approximation of the scattering phase function has been proposed.

Well–developed methods for calculating the characteristic of the brightness of the atmosphere (spherical harmonics, discrete ordinates, Monte Carlo, etc.) make it possible to take into account practically all peculiarities of the interaction of radiation with the atmosphere and the surface. Therefore, the choice of the adequate models of the atmosphere and the surface is the main problem when comparing the experimentally measured and calculated parameters of radiation.

As a rule, models for comparison of the experimental and calculated data are created based on the microphysical parameters of the atmospheric aerosol that is characteristic of the region where measurements were carried out. However, the application of microphysical models of the atmosphere with a great many variable parameters in the problems of numerical simulation, correction of aerospace images, and inverse problems of atmospheric optics, where it is important to calculate the dependences of the measured parameters on the parameters of the atmosphere, makes such calculations practically impossible. In such problems the empirical models of the atmosphere that are characterized by a few parameters are in common use.

The most complicated problems arise when the scattering phase function of the atmosphere is specified, since it can take various shapes depending on the properties of an ensemble of aerosol particles. There are two ways of parametrization of the scattering phase function. They are the adjustment of artificially created analytical functions[1–3] and the adjustment of the parameters of experimentally measured scattering phase functions.[4–6] In particular, the two–parametric model of the scattering phase function of the marine haze was given in Ref. 6. Its parameters are the extinction coefficient and air humidity.

The disadvantage of the models proposed in Refs. 5 and 6 is that they were obtained based on measurements in the climatic zone with the composition and the structure of the atmospheric aerosol being relatively constant, that is why the authors succeeded in obtaining the simple correlations of the shape of the scattering phase function with the extinction coefficient and air humidity. As pointed out below, these relations have more complicated form for the observations carried out in different regions.

Long–standing experimental measurements of the scattering phase functions in the ground atmospheric layer are carried out at the Main Geophysical Observatory (MGO). Their results were generalized in Refs. 7 and 8. In spite of the fact that a great many scattering phase functions were measured and these measurements were made in different regions, all scattering phase functions were found to be grouped together in relatively small number of classes. We use this classification for the creation of the parametric model of the scattering phase function.

The initial experimental scattering phase functions averaged over the classes and normalized by the condition

$$\frac{1}{2}\int_0^{\pi} x(\gamma)\sin\gamma\, d\gamma = 1 \tag{1}$$

are presented in Table I.

The fundamental parameter in the classification of Ref. 7 is the ratio of the portion of radiation scattered into the forward hemisphere to the portion of radiation scattered into the backward hemisphere, i.e., the elongation of the scattering phase function

$$G = \frac{\int_0^{\pi/2} x(\gamma)\sin\gamma\, d\gamma}{\int_{\pi/2}^{\pi} x(\gamma)\sin\gamma\, d\gamma}\,. \tag{2}$$

The first (before point) figure of the type of the scattering phase function indicates different values of G and is referred to as the class in accordance with notation used in Refs. 7 and 8. The scattering phase functions of neighbouring classes differ in elongation approximately 1.5 times.

TABLE I. Scattering phase functions borrowed from Refs. 7 and 8.

$\gamma°$	Type of the scattering phase function								
	1.0	2.0	3.0	3.1	4.0	4.1	5.0	5.1	6.0
1	2	3	4	5	6	7	8	9	10
0	1.500	3.575	4.957	7.540	7.277	9.421	8.872	12.204	10.931
10	1.477	3.423	4.506	6.329	6.439	7.864	7.738	6.497	9.528
20	1.412	2.739	3.502	3.944	4.568	5.276	5.341	6.129	6.527
30	1.313	2.054	2.523	2.578	3.019	3.002	3.504	3.705	3.866
40	1.190	1.559	1.790	1.663	1.961	1.884	2.189	2.193	2.241
50	1.060	1.112	1.262	1.117	1.303	1.217	1.381	1.306	1.295
60	0.938	0.875	0.919	0.826	0.903	0.814	0.891	0.805	0.771
70	0.838	0.702	0.706	0.617	0.652	0.577	0.597	0.527	0.499
80	0.773	0.615	0.581	0.525	0.511	0.442	0.443	0.336	0.353
90	0.750	0.592	0.527	0.451	0.441	0.377	0.353	0.294	0.270
100	0.773	0.592	0.515	0.473	0.413	0.349	0.314	0.260	0.233
110	0.838	0.621	0.532	0.486	0.410	0.335	0.302	0.251	0.219
120	0.938	0.677	0.573	0.503	0.428	0.368	0.307	0.266	0.219
130	1.060	0.776	0.644	0.599	0.467	0.413	0.331	0.306	0.228
140	1.190	0.938	0.745	0.719	0.534	0.490	0.374	0.396	0.241
150	1.313	1.192	0.854	0.902	0.603	0.623	0.413	0.531	0.270
160	1.412	1.458	1.002	1.147	0.701	0.811	0.466	0.725	0.307
170	1.477	1.686	1.172	1.585	0.805	1.102	0.529	1.011	0.354
180	1.500	1.762	1.236	1.637	0.843	1.232	0.555	1.137	0.367
$\gamma°$	6.1	6.2	7.0	7.1	7.2	7.3	8.0	8.1	8.2
0	13.225	26.740	14.298	14.767	22.756	24.378	17.559	18.703	22.754
10	10.751	14.276	12.153	12.167	13.506	14.539	14.366	14.752	13.990
20	6.782	6.571	7.123	6.825	7.265	7.747	7.808	7.284	7.805
30	3.894	3.624	3.886	3.978	4.035	4.003	4.017	3.912	4.201
40	2.110	1.877	2.093	2.184	2.089	2.001	1.995	2.107	2.126
50	1.204	1.069	1.178	1.200	1.103	1.010	1.047	1.077	1.083
60	0.698	0.555	0.690	0.671	0.571	0.506	0.569	0.593	0.547
70	0.433	0.308	0.430	0.400	0.305	0.253	0.341	0.341	0.275
80	0.289	0.175	0.294	0.266	0.173	0.139	0.223	0.216	0.140
90	0.220	0.116	0.220	0.195	0.112	0.085	0.166	0.151	0.086
100	0.186	0.094	0.183	0.159	0.090	0.062	0.134	0.124	0.071
110	0.176	0.098	0.168	0.146	0.090	0.062	0.124	0.108	0.080
120	0.185	0.146	0.166	0.149	0.123	0.101	0.121	0.108	0.108
130	0.214	0.175	0.174	0.172	0.160	0.121	0.125	0.122	0.115
140	0.268	0.838	0.182	0.214	0.514	0.563	0.136	0.145	0.320
150	0.357	0.640	0.207	0.283	0.396	0.393	0.153	0.194	0.253
160	0.486	0.469	0.239	0.374	0.244	0.266	0.172	0.261	0.207
170	0.685	0.387	0.269	0.512	0.211	0.222	0.190	0.378	0.183
180	0.772	0.394	0.277	0.569	0.298	0.261	0.200	0.432	0.179

$\gamma°$	8.3	8.4	9.0	9.1	9.2	9.3	10.0	10.2
0	26.898	30.968	23.276	25.670	32.020	25.252	33.147	23.538
10	15.376	16.740	18.250	19.613	17.809	15.312	21.214	15.380
20	7.746	7.732	8.173	7.952	8.369	8.309	9.281	9.005
30	4.009	3.944	3.756	3.576	3.701	4.245	3.712	4.284
40	2.050	1.966	1.772	1.708	1.780	2.084	1.326	2.078
50	1.043	0.944	0.886	0.853	0.870	0.985	0.610	0.872
60	0.519	0.409	0.463	0.478	0.449	0.476	0.292	0.417
70	0.252	0.212	0.267	0.251	0.245	0.233	0.159	0.203
80	0.130	0.103	0.168	0.148	0.137	0.113	0.095	0.110
90	0.077	0.057	0.120	0.101	0.088	0.062	0.070	0.068
100	0.056	0.042	0.094	0.080	0.064	0.043	0.060	0.051
110	0.060	0.043	0.082	0.071	0.057	0.041	0.056	0.053
120	0.083	0.074	0.078	0.071	0.061	0.061	0.053	0.059
130	0.091	0.085	0.081	0.081	0.077	0.078	0.053	0.069
140	0.407	0.487	0.087	0.095	0.150	0.243	0.056	0.132
150	0.252	0.307	0.099	0.117	0.126	0.194	0.058	0.108
160	0.187	0.232	0.112	0.147	0.088	0.155	0.064	0.086
170	0.161	0.199	0.130	0.173	0.071	0.135	0.070	0.072
180	0.174	0.210	0.135	0.183	0.076	0.132	0.073	0.065

TABLE II. Parameters of the initial scattering phase functions.

Type	1.0	2.0	3.0	3.1	4.0	4.1	5.0	5.1	6.0	6.1	6.2	7.0	7.1
G	1.0	1.46	2.08	2.14	3.13	3.43	4.76	4.58	7.22	7.06	6.76	9.65	8.96
P	1.48	1.55	1.42	1.50	1.30	1.43	1.21	1.55	1.07	1.48	8.75	1.04	1.41
Type	7.2	7.3	8.0	8.1	8.2	8.3	8.4	9.0	9.1	9.2	9.3	10.0	10.2
G	9.95	11.06	13.55	12.73	13.91	15.16	14.40	20.82	20.99	23.77	20.77	33.56	27.10
P	5.74	9.08	1.05	1.25	4.22	7.06	11.51	0.99	1.26	2.47	5.78	0.97	2.54

One parameter – elongation – is insufficient to describe the observed scattering phase functions, since the scattering phase functions of essentially different shape fall into one class. In Ref. 7 all the scattering phase functions were classified by their shape into "mildly sloping," "sharp," and "sharp with maximum." In this paper it was also noted that the mildly sloping scattering phase functions are characteristic of continental air masses, the sharp ones – for the marine air masses, and the sharp ones with maximum – for smokes and fogs. A distinctly pronounced maximum at an angle of 140°, being the maximum of the first rainbow, is typical of the last class. This allows Barteneva et al.[7] to introduce the quantitative characteristic – the ratio of the scattering phase function at an angle of 140° to that at 105°. Let us call it the sharpness of the scattering phase function, designate it by P, and extend to all scattering phase functions

$$P = \frac{2\,x(140^\circ)}{x(100^\circ) + x(110^\circ)} \,. \tag{3}$$

Let us use this value as the second parameter in classification. The second (after point) figure in Table I indicates the value of this parameter. It determines the subclass of the scattering phase function (in Table I, subclass "*0*" involves the mildly sloping scattering phase functions, subclass "*1*" – the sharp ones, and subclass "*2*" – the sharp ones with maximum. The values of elongation G and sharpness P of the initial scattering phase functions are given in Table II.

Let us note some more peculiarities of the initial scattering phase functions reported in Ref. 7. Measurement accuracy was 10% at scattering angles 20–60°, 15% at scattering angles 70–140°, and 20% at scattering angles 150–160°. The corresponding values at 0, 10, 170, and 180° were obtained by extrapolation.

The 1.0 scattering phase function is the Rayleigh scattering phase function that is naturally incorporated in the family as a limiting case.

TABLE III. Frequencies of occurence (%) of the scattering phase functions as functions of the meteorological visibility range (Refs. 7 and 8).

S, km	Class of the scattering function									
	1	2	3	4	5	6	7	8	9	10
$S > 200$	100									
$100 < S < 200$		31	67	2						
$50 < S < 100$		4	73	22	1					
				(71)	(29)					
$20 < S < 50$			3	49	45	3				
				(9)	(60)	(31)				
$10 < S < 20$				7	42	33	13			
						(75)	(25)			
$4 < S < 10$					7	21	45	23	4	
							(92)	(8)		
$2 < S < 4$						18	32	30	18	2
								(55)	(45)	
$1 < S < 2$							12	49	31	8
								(31)	(38)	(31)

The elongation of the scattering phase function is greatly correlated with the meteorological visibility range. The frequencies of occurrence of the scattering phase functions belonging to subclasses *0* (without parenthesis) and *1* (in parenthesis) are given in Table III as functions of the meteorological visibility range. The data on the scattering phase function of the subclasses *2* and more are lacking.

Each row of Table III can be considered as a histogram of the distribution of elongation of the scattering phase function for the given visibility range. These distributions are well approximated by so–called lognormal distribution. In its turn, the visibility range S can be considered as a parameter of this distribution. The following approximation is obtained from the data in Table III:

$$\rho(G) = \frac{1}{\sigma\, G\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(G) - \mu}{\sigma}\right)^2\right), \tag{4}$$

where for the scattering phase functions of subclass *0* we have $\mu = 3.041 - 0.4881\, nS$ and $\sigma = 0.340 - 1.55 \cdot 10^{-3}\, S$, whereas for the scattering phase functions of subclass *1* we have $\mu = 3.424 - 0.6181\, nS$ and $\sigma = 0.347 - 2.92 \cdot 10^{-3}\, S$. Here $\rho(G)$ is the probability density of the elongation of the scattering phase function and S is the meteorological visibility range, in kilometers.

We failed to identify the parameter determining the sharpness of the scattering phase functions. As shown in Ref. 7, correlation of P with air humidity is insignificant. Analogous conclusion was drawn in Ref. 6: correlation was observed only in the strongly turbid atmosphere (small visibility range). The authors of Refs. 6–8 concluded that the sharpness should depend on the average size of water droplets in air, but this parameter is difficult to measure. For this reason this parameter is recommended to specify depending on the types of air masses when using formula (4): subclass *0* for continental air masses and subclass *1* for marine ones.

In practice the scattering phase functions with arbitrary intermediate values of G and P observed in the real atmosphere are needed. Interpolation based on the grid of values G and P in accordance with the classification proposed in Refs. 7 and 8 is difficult because this grid is not rectangle.

Let us carry out the two–dimensional interpolation and extrapolation based on these data by a special algorithm in order to obtain the rectangle grid of the values G and P, i.e., the base table for the model scattering phase function (Table IV). The scattering phase functions $x(\gamma, G, P)$ with the parameters G and P in the rectangle domain of the base table $G_1 \le G \le G_2$, $P_1 \le P \le P_2$ are obtained as follows.

The coefficients $R_i(\gamma)$, $i = 0, 1, 2, 3$ of the interpolation formula

$$
\begin{aligned}
R_0(\gamma) &= \frac{x(\gamma,\,G_1,\,P_1)G_2\,P_2 - x(\gamma,\,G_2,\,P_1)\,G_1\,P_2 - x(\gamma,\,G_1\,P_2)\,G_2\,P_1 + x(\gamma,G_2\,P_2)\,G_1\,P_1}{(G_2 - G_1)(P_2 - P_1)}\,,\\
R_1(\gamma) &= \frac{-\,x(\gamma,\,G_1,\,P_1)\,P_2 + x(\gamma,\,G_2,\,P_1)\,P_2 + x(\gamma,\,G_1,\,P_2)\,P_1 - x(\gamma,\,G_2\,P_2)P_1}{(G_2 - G_1)(P_2 - P_1)}\,,\\
R_2(\gamma) &= \frac{-\,x(\gamma,\,G_1,\,P_1)G_2 + x(\gamma,\,G_2,\,P_1)G_1 + x(\gamma,\,G_1\,P_2)G_2 - x(\gamma,\,G_2\,P_2)G_1}{(G_2 - G_1)(P_2 - P_1)}\,,\\
R_3(\gamma) &= \frac{x(\gamma,\,G_1,\,P_1) - x(\gamma,\,G_2,\,P_1) - x(\gamma,\,G_1,\,P_2) + x(\gamma,G_2\,P_2)}{(G_2 - G_1)(P_2 - P_1)}\,.
\end{aligned}
\tag{5}
$$

are determined. Then the auxiliary parameters $r_0 - r_3$, x, and y are calculated from the formulas

$$r_i = PR_i(110°) + PR_i(100°) - 2\,R_i(140°)\,,\quad i = 0,\,\ldots,\,3, \tag{6}$$

$$x = \frac{G(1 + G_1 + G_2) - G_1G_2}{G + 1}\,,$$

$$y = \frac{r_1G_2G_1 - r_0(G + 1) - r_1G(1 + G_1 + G_2)}{r_2(G + 1) + r_3G(1 + G_1 + G_2) - r_3G_1G_2}\,. \tag{7}$$

The sought–for scattering phase function is finally calculated from the formula

$$x(\gamma,\,G,\,P) = R_0(\gamma) + R_1(\gamma)\,x + R_2(\gamma)\,y + R_3(\gamma)\,xy\,. \tag{8}$$

Table IV and formulas (5)–(8) yield the model scattering phase function with two parameters: the elongation G and the sharpness P. We note that the spherical ($G = 1$, $P = 1$) and Rayleigh ($G = 1$, $P = 1.48$) scattering phase functions are the particular cases of our model.

Let us determine the accuracy of approximation of the experimental scattering phase functions by this model. The values of the mean square deviation of the model scattering phase functions from the experimental ones (in %) are given in the column "Model 1" of Table V along with the maximum deviation and the angle at which the deviation is maximum.

Since the experimental scattering phase functions at the angles of 0, 10, 170, and 180° were obtained by extrapolation, we did not compare the results at these angles when testing the models.

An analysis of the results presented here shows that the model approximates well the experimental scattering phase functions. Mean and even maximum errors in approximating do not exceed the measurement error for the majority of scattering phase functions.

The interpolation from the table creates no problems when using a computer. However, analytical approximation of the scattering phase function is more preferable in many theoretical problems, in particular, in the expansion into a series in the Legendre polynomials that are necessary for solving the problems of radiative transfer by the method of spherical harmonics.

TABLE IV. Base table of the model scattering phase function.

γ°	G = 1.0						G = 1.5			
	P = 1.0	P = 1.25	P = 1.5	P = 2.3	P = 3.5	P = 5.3	P = 1.0	P = 1.25	P = 1.5	P = 2.3
1	2	3	4	5	6	7	8	9	10	11
0	1.000	1.288	1.518	2.018	2.433	2.758	3.349	3.347	3.637	5.458
10	1.000	1.275	1.495	1.972	2.368	2.679	2.986	3.070	3.344	4.783
20	1.000	1.238	1.427	1.840	2.182	2.450	2.290	2.482	2.670	3.432
30	1.000	1.180	1.324	1.636	1.896	2.099	1.755	1.909	2.005	2.093
40	1.000	1.110	1.197	1.387	1.545	1.669	1.361	1.473	1.532	1.531
50	1.000	1.035	1.062	1.122	1.172	1.211	1.133	1.165	1.172	1.116
60	1.000	0.964	0.935	0.873	0.821	0.780	0.984	0.952	0.917	0.805
70	1.000	0.906	0.832	0.670	0.535	0.430	0.887	0.814	0.750	0.591
80	1.000	0.869	0.764	0.537	0.348	0.201	0.841	0.734	0.646	0.452
90	1.000	0.856	0.741	0.491	0.283	0.121	0.814	0.699	0.607	0.405
100	1.000	0.869	0.764	0.537	0.348	0.201	0.801	0.694	0.608	0.423
110	1.000	0.906	0.832	0.670	0.535	0.430	0.788	0.712	0.648	0.499
120	1.000	0.964	0.935	0.873	0.821	0.780	0.792	0.755	0.724	0.653
130	1.000	1.035	1.062	1.122	1.172	1.211	0.792	0.810	0.822	0.840
140	1.000	1.110	1.197	1.387	1.545	1.669	0.794	0.879	0.943	1.060
150	1.000	1.180	1.324	1.636	1.896	2.099	0.785	0.943	1.067	1.323
160	1.000	1.238	1.427	1.840	2.182	2.450	0.803	1.024	1.205	1.607
170	1.000	1.275	1.493	1.972	2.368	2.679	0.892	1.121	1.337	1.935
180	1.000	1.288	1.518	2.018	2.433	2.758	0.917	1.159	1.397	2.078

TABLE IV continued.

$\gamma°$	$G = 1.5$			$G = 2.3$						
	$P = 3.5$	$P = 5.3$	$P = 8.0$	$P = 1.0$	$P = 1.25$	$P = 1.5$	$P = 2.3$	$P = 3.5$	$P = 5.3$	$P = 8.0$
0	6.889	7.944	8.666	5.993	5.809	6.219	9.177	11.31	12.81	13.84
10	5.914	6.748	7.321	5.268	5.184	5.528	7.789	9.421	10.57	11.35
20	4.007	4.434	4.731	3.596	3.780	4.046	5.058	5.787	6.299	6.651
30	2.163	2.218	2.262	2.497	2.589	2.615	2.488	2.397	2.333	2.290
40	1.531	1.533	1.539	1.661	1.772	1.799	1.623	1.496	1.408	1.347
50	1.071	1.039	1.020	1.203	1.247	1.241	1.079	0.962	0.880	0.824
60	0.716	0.650	0.604	0.946	0.920	0.879	0.722	0.608	0.529	0.474
70	0.466	0.373	0.306	0.772	0.715	0.660	0.507	0.397	0.320	0.267
80	0.299	0.184	0.102	0.689	0.601	0.528	0.367	0.251	0.169	0.113
90	0.247	0.128	0.0411	0.633	0.545	0.474	0.322	0.212	0.134	0.0803
100	0.278	0.169	0.0893	0.611	0.525	0.456	0.314	0.211	0.139	0.0884
110	0.381	0.293	0.229	0.599	0.527	0.467	0.331	0.233	0.164	0.116
120	0.598	0.557	0.528	0.593	0.552	0.518	0.441	0.386	0.347	0.320
130	0.855	0.866	0.875	0.595	0.593	0.588	0.567	0.552	0.541	0.534
140	1.153	1.223	1.274	0.605	0.657	0.692	0.741	0.777	0.802	0.819
150	1.524	1.674	1.784	0.603	0.725	0.822	1.025	1.171	1.273	1.344
160	1.923	2.160	2.330	0.594	0.817	0.999	1.397	1.684	1.885	2.024
170	2.405	2.755	3.003	0.576	0.919	1.218	1.939	2.459	2.824	3.075
180	2.614	3.012	3.293	0.547	0.961	1.320	2.185	2.810	3.248	3.549

$\gamma°$	$G = 2.3$	$G = 3.5$							$G = 5.3$
	$P = 12$	$P = 1.25$	$P = 1.5$	$P = 2.3$	$P = 3.5$	$P = 5.3$	$P = 8.0$	$P = 12$	$P = 1.25$
0	14.53	8.086	9.956	13.14	15.47	17.16	18.49	19.49	10.24
10	11.89	7.000	8.035	10.28	11.91	13.08	13.96	14.60	8.484
20	6.889	4.865	5.428	6.245	6.843	7.251	7.498	7.663	5.851
30	2.261	3.147	3.064	2.864	2.717	2.615	2.545	2.496	3.648
40	1.307	2.011	1.901	1.691	1.538	1.430	1.353	1.299	2.183
50	0.786	1.312	1.215	1.050	0.930	0.846	0.789	0.750	1.330
60	0.437	0.888	0.802	0.661	0.557	0.485	0.435	0.400	0.830
70	0.230	0.630	0.561	0.448	0.365	0.307	0.268	0.240	0.546
80	0.0746	0.492	0.421	0.313	0.234	0.180	0.141	0.118	0.393
90	0.0439	0.415	0.358	0.265	0.197	0.149	0.117	0.0943	0.310
100	0.0544	0.385	0.329	0.238	0.172	0.126	0.0941	0.0719	0.274
110	0.0847	0.376	0.315	0.215	0.141	0.0900	0.0556	0.0316	0.262
120	0.302	0.391	0.349	0.282	0.232	0.198	0.175	0.158	0.268
130	0.529	0.422	0.395	0.352	0.320	0.297	0.281	0.269	0.291
140	0.832	0.476	0.483	0.521	0.548	0.571	0.599	0.621	0.335
150	1.393	0.537	0.621	0.763	0.866	0.939	0.991	1.027	0.388
160	2.118	0.622	0.818	1.119	1.340	1.491	1.584	1.646	0.463
170	3.245	0.732	1.123	1.715	2.147	2.443	2.619	2.734	0.564
180	3.752	0.778	1.259	1.987	2.520	2.882	3.097	3.238	0.606

$\gamma°$	$G = 5.3$						$G = 8.0$			
	$P = 1.5$	$P = 2.3$	$P = 3.5$	$P = 5.3$	$P = 8.0$	$P = 12$	$P = 1.0$	$P = 1.25$	$P = 1.5$	$P = 2.3$
0	12.11	15.10	18.40	21.78	24.97	27.73	12.01	13.66	14.78	17.06
10	8.565	9.922	11.42	12.50	14.38	15.62	10.47	11.34	11.83	12.29
20	6.193	6.205	6.218	6.231	6.243	6.254	6.808	6.817	6.831	6.904
30	3.724	3.665	3.600	3.534	3.472	3.418	3.881	3.901	3.912	3.917
40	2.155	2.071	1.979	1.884	1.796	1.720	2.180	2.158	2.143	2.113
50	1.281	1.232	1.177	1.122	1.069	1.024	1.251	1.210	1.187	1.163
60	0.780	0.731	0.676	0.620	0.568	0.523	0.745	0.700	0.676	0.648
70	0.504	0.465	0.421	0.376	0.334	0.297	0.477	0.434	0.409	0.383
80	0.341	0.310	0.276	0.240	0.207	0.178	0.337	0.295	0.271	0.245
90	0.276	0.245	0.210	0.174	0.140	0.111	0.257	0.221	0.201	0.178
100	0.242	0.211	0.178	0.143	0.111	0.0830	0.219	0.187	0.168	0.148
110	0.232	0.203	0.171	0.139	0.108	0.0817	0.205	0.174	0.156	0.140
120	0.243	0.222	0.198	0.175	0.152	0.133	0.203	0.177	0.163	0.154
130	0.274	0.254	0.232	0.209	0.188	0.170	0.208	0.194	0.187	0.180
140	0.355	0.476	0.610	0.747	0.876	0.988	0.212	0.225	0.243	0.331
150	0.439	0.503	0.572	0.644	0.711	0.769	0.228	0.280	0.311	0.349
160	0.563	0.570	0.577	0.584	0.590	0.595	0.244	0.357	0.410	0.387
170	0.750	0.710	0.666	0.621	0.577	0.539	0.252	0.471	0.570	0.496
180	0.832	0.779	0.721	0.661	0.604	0.553	0.248	0.518	0.641	0.561

TABLE IV continued.

$\gamma°$	G = 8.0				G = 12				
	P = 3.5	P = 5.3	P = 8.0	P = 12	P = 1.0	P = 1.25	P = 1.5	P = 2.3	P = 3.5
0	19.69	22.55	25.51	28.14	16.49	18.75	19.97	20.91	22.02
10	12.83	13.49	14.38	15.17	13.73	14.87	15.36	14.93	14.43
20	6.988	7.069	7.126	7.177	7.706	7.307	7.137	7.326	7.548
30	3.923	3.912	3.857	3.808	3.979	3.864	3.821	3.930	4.058
40	2.078	2.036	1.980	1.931	1.996	2.050	2.077	2.087	2.099
50	1.136	1.103	1.062	1.026	1.077	1.070	1.067	1.068	1.070
60	0.616	0.580	0.543	0.509	0.598	0.597	0.594	0.578	0.558
70	0.352	0.319	0.286	0.257	0.368	0.345	0.332	0.315	0.295
80	0.216	0.186	0.159	0.135	0.247	0.221	0.206	0.188	0.166
90	0.152	0.126	0.101	0.0800	0.186	0.157	0.142	0.126	0.107
100	0.126	0.102	0.0793	0.0592	0.151	0.129	0.116	0.103	0.0866
110	0.122	0.102	0.0814	0.0630	0.140	0.115	0.101	0.0934	0.0840
120	0.145	0.134	0.120	0.108	0.136	0.115	0.105	0.104	0.104
130	0.173	0.163	0.148	0.134	0.139	0.129	0.124	0.124	0.124
140	0.433	0.541	0.643	0.733	0.146	0.152	0.163	0.225	0.299
150	0.392	0.437	0.477	0.513	0.158	0.197	0.219	0.241	0.266
160	0.362	0.340	0.333	0.327	0.171	0.258	0.296	0.270	0.238
170	0.411	0.331	0.281	0.237	0.175	0.353	0.429	0.353	0.264
180	0.470	0.381	0.320	0.265	0.176	0.395	0.490	0.403	0.301

$\gamma°$	G = 12			G = 18				
	P = 5.3	P = 8.0	P = 12	P = 1.0	P = 1.25	P = 1.5	P = 2.3	P = 3.5
0	23.66	24.07	18.41	21.94	24.23	25.38	25.34	25.30
10	14.28	14.64	13.47	17.35	18.62	19.09	17.82	16.38
20	7.724	7.999	8.869	8.109	7.824	7.705	7.892	8.104
30	4.123	4.122	4.453	3.819	3.647	3.586	3.791	4.023
40	2.093	2.038	2.018	1.819	1.789	1.785	1.878	1.982
50	1.063	1.000	0.873	0.922	0.898	0.891	0.922	0.958
60	0.537	0.497	0.426	0.486	0.501	0.507	0.499	0.490
70	0.274	0.248	0.220	0.284	0.269	0.261	0.255	0.247
80	0.146	0.131	0.116	0.181	0.162	0.151	0.141	0.129
90	0.0899	0.0773	0.0639	0.131	0.111	0.100	0.0898	0.0781
100	0.0704	0.0555	0.0413	0.103	0.0891	0.0805	0.0704	0.0590
110	0.0731	0.0537	0.0257	0.0921	0.0786	0.0709	0.0635	0.0551
120	0.100	0.0862	0.0737	0.0882	0.0786	0.0735	0.0718	0.0698
130	0.119	0.109	0.124	0.0911	0.0892	0.0882	0.0874	0.0866
140	0.380	0.437	0.401	0.0978	0.105	0.114	0.154	0.200
150	0.289	0.325	0.422	0.110	0.132	0.145	0.163	0.182
160	0.211	0.224	0.277	0.122	0.169	0.191	0.179	0.166
170	0.189	0.187	0.219	0.137	0.213	0.246	0.209	0.167
180	0.216	0.215	0.249	0.141	0.231	0.271	0.226	0.174

$\gamma°$	G = 18			G = 27				
	P = 5.3	P = 8.0	P = 12	P = 1.0	P = 1.25	P = 1.5	P = 2.3	P = 3.5
0	25.00	22.42	16.97	30.08	29.61	29.28	28.60	27.90
10	15.19	14.45	13.06	17.32	20.48	21.84	19.85	17.81
20	8.267	8.454	8.998	9.774	8.727	8.210	8.364	8.522
30	4.231	4.434	4.777	4.237	3.665	3.409	3.670	3.978
40	2.081	2.155	2.216	1.576	1.524	1.521	1.696	1.877
50	0.991	0.991	0.940	0.672	0.720	0.752	0.810	0.870
60	0.483	0.471	0.436	0.256	0.377	0.439	0.438	0.437
70	0.239	0.225	0.210	0.168	0.195	0.209	0.212	0.215
80	0.118	0.105	0.0949	0.112	0.113	0.113	0.110	0.107
90	0.0660	0.0540	0.0433	0.0844	0.0782	0.0741	0.0676	0.0610
100	0.0473	0.0355	0.0255	0.0733	0.0632	0.0569	0.0491	0.0411
110	0.0459	0.0320	0.0139	0.0663	0.0587	0.0536	0.0442	0.0347
120	0.0665	0.0593	0.0530	0.0612	0.0581	0.0558	0.0502	0.0445
130	0.0837	0.0817	0.0947	0.0562	0.0632	0.0666	0.0645	0.0623
140	0.247	0.270	0.237	0.0698	0.0762	0.0829	0.107	0.133
150	0.201	0.231	0.296	0.0639	0.0811	0.0922	0.109	0.126
160	0.158	0.180	0.222	0.0614	0.0902	0.105	0.108	0.111
170	0.138	0.157	0.189	0.111	0.0896	0.0794	0.0856	0.0921
180	0.138	0.157	0.192	0.135	0.0884	0.0658	0.0738	0.0820

TABLE IV continued.

$\gamma°$	G = 27			G = 40			
	P = 5.3	P = 8.0	P = 12	P = 3.5	P = 5.3	P = 8.0	P = 12
0	26.98	25.59	25.59	29.52	29.07	29.87	30.43
10	15.92	14.42	14.42	18.59	16.48	14.39	12.71
20	8.662	8.759	8.759	8.801	8.921	9.071	9.191
30	4.256	4.499	4.499	3.969	4.233	4.450	4.630
40	2.049	2.196	2.196	1.824	2.014	2.180	2.315
50	0.926	0.973	0.973	0.817	0.879	0.936	0.983
60	0.437	0.438	0.438	0.403	0.404	0.402	0.401
70	0.217	0.217	0.217	0.195	0.204	0.215	0.224
80	0.104	0.0993	0.0993	0.0929	0.0967	0.103	0.108
90	0.0541	0.0470	0.0470	0.0498	0.0481	0.0493	0.0499
100	0.0333	0.0260	0.0260	0.0293	0.0250	0.0227	0.0205
110	0.0254	0.0170	0.0170	0.0205	0.0123	0.00603	0.000780
120	0.0390	0.0344	0.0344	0.0270	0.0202	0.0142	0.00935
130	0.0602	0.0583	0.0583	0.0459	0.0434	0.0415	0.0399
140	0.155	0.172	0.172	0.0871	0.0991	0.115	0.128
150	0.143	0.158	0.158	0.0895	0.102	0.110	0.117
160	0.116	0.127	0.127	0.0765	0.0826	0.0767	0.0716
170	0.102	0.118	0.118	0.0478	0.0729	0.0801	0.0864
180	0.0929	0.110	0.110	0.0277	0.0608	0.0766	0.0895

TABLE V. Results of comparison of experimentally measured and model scattering phase functions. Here md denotes mean deviation of the model from the experimental scattering phase function, in %; max denotes maximum deviation of the model from the experimental scattering phase function, in %; γ_{max} *indicates the angle of maximum deviation, in deg.*

type	Model 1			Model 2			Model 3			Model 4		
	md	max	$\gamma°_{max}$	md	max	$\gamma°_{max}$	md	max	$\gamma°_{max}$	md	max	$\gamma°_{max}$
spherical	0	0	–	0	0	–	0	0	–	0	0	–
1.0	0	0	–	2.7	7.4	20	0	0	–	0	0	–
2.0	5.1	14	160	4.3	11	70	13	29	20	11	22	20
3.0	1.6	4.8	20	6.0	14	160	14	39	70	9.6	22	60
3.1	5.7	10	50	7.4	18	70	22	56	70	18	36	50
4.0	1.8	3.1	20	7.0	28	160	21	59	70	12	30	60
4.1	1.2	4.2	160	5.9	23	160	27	75	70	19	41	60
5.0	2.4	4.1	60	15	53	160	27	77	70	12	31	60
5.1	3.8	10	160	22	37	160	32	93	80	21	38	60
6.0	2.7	5.7	160	21	65	160	38	110	70	16	48	60
6.1	1.9	8.1	160	46	59	50	41	124	70	24	54	60
6.2	6.1	12	120	–	–	–	–	–	–	–	–	–
7.0	2.0	5.5	160	30	70	160	46	136	70	21	58	60
7.1	2.3	4.3	160	61	75	50	43	138	70	24	55	60
7.2	7.3	16	130	–	–	–	–	–	–	–	–	–
7.3	7.3	21	110	–	–	–	–	–	–	–	–	–
8.0	2.4	4.1	110	51	64	40	60	180	70	25	79	60
8.1	2.3	6.9	160	78	87	40	54	172	70	29	71	50
8.2	5.6	18	120	–	–	–	–	–	–	–	–	–
8.3	9.1	25	110	–	–	–	–	–	–	–	–	–
8.4	15	50	110	–	–	–	–	–	–	–	–	–
9.0	6.5	16	60	74	85	40	75	229	70	33	103	50
9.1	3.6	7.2	160	–	–	–	75	231	70	37	110	50
9.2	11	45	160	–	–	–	70	180	50	46	102	50
9.3	5.3	15	110	–	–	–	–	–	–	–	–	–
10.0	17	52	60	–	–	–	125	389	70	53	182	50
10.2	10	26	160	–	–	–	80	214	70	43	99	50

There is a number of analytical approximations of the scattering phase functions,[1–3] particularly, the Henyey–Greenstein formula[1] that can be very simply expanded into a series in the Legendre polynomials (coefficient at the ith order polynomial is $x_i = (2i + 1)g^i/2$). The known approximation expressions for the scattering phase functions can be always generalized and refined by introducing the additional parameter, for example, by raising the approximation formula to the power α, or by representing the scattering phase function as a sum (with coefficients) of two initial scattering phase functions elongated in the forward and backward directions.

Let us use the second practice for the Henyey–Greenstein scattering phase function[1] and for the binomial one.[2] We obtain based on the Henyey–Greenstein scattering phase function

$$x(\gamma) = a\frac{1 - g^2}{(1 + g^2 - 2g\cos\gamma)^{3/2}} + (1 - a)\frac{1 - g^2}{(1 + g^2 + 2g\cos\gamma)^{3/2}}. \quad (9)$$

This scattering phase function can be easy expanded in the Legendre polynomials, with the coefficient at the ith order polynomial in the form

$$x_i = \begin{cases} \dfrac{2i + 1}{2} g^i, & \text{for even } i, \\ \dfrac{2i + 1}{2}(2a - 1) g^i, & \text{for odd } i. \end{cases} \quad (10)$$

We analogously obtain on the basis of the binomial scattering phase function[2]

$$x(\gamma) = \frac{N + 1}{2^N}\left(a(1 + \cos\gamma)^N + (1 - a)(1 - \cos\gamma)^N\right). \quad (11)$$

The expansion of scattering phase function (11) in the Legendre polynomials is given by the recursion relation

$$x_i = \begin{cases} c_i, & \text{for even } i, \\ (2a - 1)c_i, & \text{for odd } i, \end{cases} \quad (12)$$

where

$$c_i = \frac{N + 1 - i}{N + 1 + i} c_{i-1}, \quad c_0 = 1.$$

The results of approximation of the initial experimental phase functions by formulas (9) and (11) are given in Table V in columns "Model 2" and "Model 3". Dashes drawn in the table indicate the lack of the scattering phase functions (9) or (11) with appropriate G and P. An analysis of the data in this table shows that the binomial scattering phase function approximates the real scattering phase functions worse than the Henyey–Greenstein one.

The function

$$x(\gamma) = (a + b\cos^2(\gamma)) \exp(\alpha\cos\gamma) \quad (13)$$

can be considered as another approximation. It has quite simple form, and the spherical and Rayleigh scattering phase functions are its particular cases.

The values of the coefficients a, b, and α are determined from normalization condition (1) and the conditions of the equality of the elongation and the sharpness of scattering phase function (13) to the corresponding values of G and P [expressions (2) and (3)]. Thus we have a system of three equations in three unknowns for determining the parameters. Solving it for $G = 1$, we obtain

$$\alpha = 0, \qquad a = \frac{3.521 - 0.411P}{1.521 + 1.559P}, \qquad b = 3(1 - a). \quad (14)$$

For $G > 1$, we have

$$a = \frac{4(1 + G) - e^{\alpha}(2\alpha^2 - 4\alpha + 4) - Ge^{-a}(2\alpha^2 + 4\alpha + 4)}{(1 + G)(e^{-\alpha}(2 + \alpha) + e^{\alpha}(2 - \alpha) - 4)},$$

$$b = 2\alpha^2 \frac{e^{\alpha} + Ge^{-\alpha} - 1 - G}{(1 + G)(e^{-\alpha}(2 + a) + e^{\alpha}(2 - a) - 4)}. \quad (15)$$

In order to obtain the equation in the unknown α, we must substitute expressions (15) into the following equality:

$$2a + 1.174b - P(a + 0.03b)\exp(0.5924\alpha) - $$

$$- P(a + 0.117b)\exp(0.424\alpha) = 0. \quad (16)$$

Equation (16) can be easy solved on a computer by the branching technique (bisection) in the interval [0, 3].

Analytical expressions for the coefficients of the expansion of scattering phase function (13) in the Legendre polynomials are not so simple as that for the Henyey––Greenstein scattering phase function, but they can be easy calculated on a computer. These expressions can be written in the simplest form using the recursion relations

$$x_i = \frac{2i + 1}{2}\sum_{k=0}^{i} C_k(i)(aI_k + bI_{k+2}), \quad (17)$$

where

$$c_k(i) = \frac{2i - 1}{i} c_{k-1}(i - 1) - \frac{i - 1}{i} C_k(i - 2),$$

$$C_{-1}(i - 1) = C_{i-1}(i - 2) = C_i(i - 2) = 0,$$

$$C_0(0) = 1, \; C_0(1) = 0, \; C_1(1) = 1,$$

$$I_k = \begin{cases} \dfrac{2}{\alpha}\sinh\alpha - \dfrac{k}{\alpha} i_{k-1}, & \text{for even } k, \\ \dfrac{2}{\alpha}\cosh\alpha - \dfrac{k}{\alpha} i_{k-1}, & \text{for odd } k, \end{cases} \qquad I_{-1} = 0.$$

The results of approximation of the initial experimental scattering phase functions by formula (13) are given in Table V in the column "Model 4". Dashes drawn in Table V indicate that the scattering phase function obtained by formula (13) has no physical sense (negative values are obtained).

In most cases scattering phase function (13) approximates the experimental scattering phase functions better than the Henyey––Greenstein one, and can be recommended as an analytical expression for modelling scattering phase functions in the atmosphere.

Thus, we propose the model scattering phase function with two parameters: sharpness and elongation. These parameters have a physical sense. Elongation is the ratio of the portion of radiation scattered in the forward direction to the portion of radiation scattered in the backward direction. Sharpness describes the relative magnitude of the maximum of the first rainbow. Two model scattering phase functions based on the prescribed values of sharpness and elongation have been proposed: interpolation model (Table IV and formulas (5)–(8)) and analytical model [formulas (13)–(16)]. The choice from these models can be based on the technique of calculation of the radiative transfer and on the desired accuracy of approximation of the real scattering phase functions.

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