INTEGRAL THEOREMS ON OPTICAL REFRACTION IN A THREE-DIMENSIONALLY NONUNIFORM ATMOSPHERE AND THEIR GEODETIC APPLICATIONS

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Analytical relations between the integral characteristics of geometric-optics rays and the parameters of the three-dimensionally nonuniform atmosphere of the earth are derived. Modified laws of refraction of light and new methods for determining atmospheric corrections to the range are derived with the help of these relations.

The geometric-optics approximation is now widely employed for calculating atmospheric corrections to optical measurements of range or angular dimensions. On the basis of this approximation the atmospheric corrections are determined by integrals over the trajectories of geometric-optic rays, to find which the ray equations must be solved.[1]

Method for calculating the atmospheric corrections are discussed in Refs. 1–3. In these methods it is necessary to know the true spatial distribution of the index of refraction of air (which is not always possible in practice) or some approximate model profile is substituted for this distribution (in doing so the atmospheric corrections are determined with a lower accuracy). A new approach to this problem is proposed in Refs. 4 and 5. In this approach the atmospheric corrections are determined by methods in which there is no need to use the true or model atmospheric profile and the explicit form of the ray trajectories need not be sought. The approach essentially consists of the fact that the standard[1] differential equations for the rays are replaced by equivalent integral relations for some quantities averaged along the rays.[4,5] In addition, these quantities are chosen so that they can be represented directly in terms of experimentally determined quantities.

We shall use the approach of Ref. 5 to examine new formulations of the laws of refraction of geometric-optics rays in a three-dimensionally nonuniform atmosphere and new methods for determining the atmospheric corrections to the range.

Following Ref. 5, we shall choose as the starting relations the first integral of the ray equation of geometric optics

$$n_L\vec{l}_L - n_0\vec{l}_0 = \int_0^{\sigma_L} \vec{\nabla} n\left[r(\sigma)\right]\, d\sigma, \tag{1}$$

as well as the index of refraction of air averaged along the trajectory of a geometric-optics ray

$$\bar{n} = S^{-1}\int_0^{\sigma_L} n\left[r(\sigma)\right]\, d\sigma, \tag{2}$$

and the radius vector $\vec{r}_L$ connecting the starting point ($\vec{r} = 0$) of the ray trajectory and the final point ($\vec{r} = \vec{r}_L$),

$$r_L = \int_0^{\sigma_L} \vec{l}(\sigma)\, d\sigma. \tag{3}$$

Here $n(\vec{r})$ is the three-dimensionally nonuniform profile of the index of refraction of air; σ is the ray coordinate measured along the trajectory of the ray (σ = 0 at $\vec{r} = 0$); $\vec{l} = \dfrac{d\vec{r}}{d\sigma}$ is the unit vector tangent to the ray; $S = \int_{\sigma}^{\sigma_L} d\sigma$ is the length of the trajectory of the ray; $L = |\vec{r}_L|$ is the distance between the end points of the ray trajectory along a straight line; the indices 0 and L denote quantities determined at the starting and end points of the trajectory, respectively.

Representing the integrals (1)–(3) in the form of an Euler-Maclaurin expansion[6] we obtain

$$n_L\vec{l}_L - n_0\vec{l}_0 = \frac{\vec{\nabla} n_L + \vec{\nabla} n_0}{2}, \tag{4}$$

$$\bar{n} = \frac{n_0 + n_L}{2} - S/12\left(n'_L - n'_0 \right), \tag{5}$$

$$r_L = \frac{\vec{l}_0 + \vec{l}_L}{2} S - S^2/12\left(\vec{l}'_L - \vec{l}'_0 \right), \tag{6}$$

where

$$\vec{\nabla} n = n'\vec{l} + n\vec{l}', \tag{7}$$

the notation $n' \equiv \dfrac{dn}{d\sigma'}$, $\vec{l}' = \dfrac{d\vec{l}}{d\sigma}$ was introduced, and the terms were proportional to $\dfrac{d^3\vec{n}}{d\sigma^3}$, $\dfrac{d^3 l}{d\sigma^3}$, and derivatives of higher order were dropped.[5]

We shall study several specific examples of the application of the equations of geometric optics for the ray-averaged quantities (4)–(7).

1. We shall derive Snell's law of refraction for a plane-layered medium. Let the properties of the medium vary in the direction $\vec{n}$. Then $\vec{\nabla} n_L + \vec{\nabla} n_0 = g\vec{h}$ and formula (4) can be represented in the form

$$n_L \vec{l}_L + n_0 \vec{l}_0 = \frac{gS}{2}\vec{h}. \tag{8}$$

Multiplying Eq. (8) alternately by $\vec{h}$ and $\vec{l}_L$ and eliminating $gS/2$ from the equations obtained we arrive at the relation

$$n_L - n_0 \cos(\alpha_0 + \alpha_L) = [n_L \cos z_{vL} - n_0 \cos z_{v0}] \cos z_{vL}$$

where

$$\cos(\alpha_0 + \alpha_L) = (\vec{l}_0 \vec{l}_L);\quad \cos z_{V0} = (\vec{h}\vec{l}_0);\quad \cos z_{VL} = (\vec{h}\vec{l}_L);$$

α_0 and α_L are the angles of refraction; and, z_{v0} and z_{vL} are the visible, refraction-distorted, zenith angles.

Since $\alpha_0 + \alpha_L = z_{vL} - z_{v0}$ we have

$$n_0 \sin z_{v0} = n_L \sin z_{vL} \quad — \tag{9}$$

the well-known law of refraction for a plane-layered medium. Analysis shows that with the help of Eqs. (4)–(7) it is not difficult to extend the law of refraction to the case of a three-dimensionally nonuniform medium (one mathematical formulation of such a generalized law will be presented below when we examine the refraction correction to the range).

2. We shall now modify the Laplace-Oriani theorem for the case of a three-dimensionally nonuniform medium. For the conditions under which this theorem is true ($\vec{l}_L = \dfrac{\vec{r}_L}{L}$, $n_L = 1$ and $\vec{\nabla} n_L = 0$) the formula (4) assumes the form

$$\frac{r_L}{L} - n_0 \vec{l}_0 = \frac{\vec{\nabla} n_0}{2} S. \tag{10}$$

Multiplying Eq. (10) alternating by $\vec{l}_0$ and $\vec{h}_0$ ($\vec{h}_0$ points toward the zenith at the starting point of the trajectory) we obtain

$$\cos\alpha_0 - n_0 = \frac{g_{v0} \cos z_{v0} + g_{g0} \sin z_{v0}}{2} \cdot S, \tag{11}$$

$$\cos z_t - n_0 \cos z_{v0} = \frac{g_{v0} S}{2}, \tag{12}$$

where z_t is the true zenith angle at the point of observation; g_{v0} is the projection of $\vec{\nabla} n_0$ on $\vec{h}_0$; and, g_{h0} is the horizontal projection of $\vec{\nabla} n_0$ (the projection on the axis perpendicular to $\vec{h}_0$ and lying in a plane passing through the vectors $\vec{h}_0$ and $\vec{l}_0$).

Dropping further the index 0 and using the fact that

$$\cos \alpha_{lat} = \frac{\cos \alpha - \cos z_v \cdot \cos z_t}{\sin z_t \cdot \sin z_v}$$

where α_{lat} is the angle of lateral refraction, we obtain from Eqs. (11) and (12)

$$\left[\frac{\cos\alpha \cdot \cos z_v - \cos z_{lat} \cdot \sin z_v \cdot \sin\alpha}{\cos^2\alpha_{lat} \cdot \sin^2 z_v + \cos^2 z_v} \sqrt{1 - \sin^2 z_v \frac{\sin^2\alpha_{lat}}{\sin^2\alpha}} - n_0 \cos z_v\right] \tag{13}$$

Equation (13) relates the angle of refraction α with the visible zenith angle z_v, the angle of lateral refraction α_{lat}, the index of refraction n_0, and the vertical and horizontal gradients of the index of refraction at the point of observation, and it extends the Laplace-Oriani theorem to the case of a three-dimensionally nonuniform medium. In the case of lateral refraction $\alpha_{lat} = 0$ we obtain from Eq. (13)

$$\alpha = (n_0 - 1) \cdot \frac{\sin z_v - \dfrac{g_h}{g_v} \cos z_v}{\cos z_v + \dfrac{g_h}{g_v} \sin z_v}, \tag{14}$$

whence, if the condition $g_h \le g_v$ is satisfied, it follows that

$$\alpha = (n_0 - 1) \cdot \mathrm{tg} z_v - (n_0 - 1) \cdot \left[1 + \mathrm{tg}^2 z_v\right] \frac{g_h}{g_v}. \tag{15}$$

The first term in Eq. (15) gives the well-known[2] relation of the Laplace-Oriani theorem for a plane-layered medium, while the second term describes the change introduced by the horizontal nonuniformity in the refraction.[2]

3. The relation between L and S in a three-dimensionally nonuniform medium and the relation for the atmospheric correction $\delta L = S - L$, taking into account the refraction-induced curvature of the ray, can be derived with the help of Eqs. (4)–(7):

$$(r_L \vec{l}_L)\, n_0 + (r_L \vec{l}_0)\, n_L = \frac{S}{6}(n_L + \vec{n}_0) + \frac{5S}{6}(n_L + \vec{n}_0) \cdot (\vec{l}_L \vec{l}_0) + \vec{n} S \left[1 - (\vec{l}_L \vec{l}_0)\right], \tag{16}$$

$$\delta L = S\left\{1 - \frac{[1+5(\vec{l}_0\vec{l}_L)]+6\bar{n}[1-(\vec{l}_0\vec{l}_L)]}{6[n_0\cos\alpha_L + n_L\cos\alpha_0]}(n_0+n_L)\right\} \quad (17)$$

where $\bar{n}$ is given by the formula (5).

We note that the formula (17) follows from the relation (16), which describes the change in the direction of the ray as it propagates between the points $\vec{r} = 0$ and $\vec{r} = \vec{r}_L$, and it is the law of refraction of a three-dimensionally nonuniform medium.

To determine the correction δL which the help of Eq. (17) there is no need to know the profile of the index of refraction on the measurement path and it is not necessary to use any a priori atmospheric models. All quantities appearing in Eq. (17) can be determined by performing measurements at the end points of the path.

4. We shall now study the relation for the index of refraction of air averaged along the ray trajectory (5). The quantity n takes into account the fact that the velocity of propagation of an optical signal in the atmosphere differs from the velocity of light in a vacuum and it is the main correction accounting for the effect of the earth's atmosphere on the accuracy of range measurements.[3] Extending the formula (5) to the case in which the values of the index of refraction not only at the end points (n_0, n_L) but also at a series of intermediate points of the path (n_i, $i = 1, 2, \ldots$) are employed in the Euler—Maclaurin expansion, we obtain[6,8]

$$\bar{n} = n_p - \frac{S}{12N^2}\cdot\left[\left(\vec{l}_L\vec{\nabla}n_L\right) - \left(\vec{l}_0\vec{\nabla}n_0\right)\right], \quad (18)$$

where

$$n_p = \frac{1}{N}\left[\frac{n_0 + n_L}{2} + \sum_{i=1}^{N-1} n_i\right], \quad (19)$$

$$\left(\vec{l}\vec{\nabla}n\right) = g_v\cdot\cos z_v + g_h\cdot\sin z_v, \quad (20)$$

where n_p is the result of the point approximation of the mean integral index of refraction by the trapezoidal method; $N - 1$ is the number of intermediate points along the path at which measurements of n_1 are performed; g_v and g_h are the vertical and horizontal components of $\vec{\nabla}n$; and, z_v is the visible zenith angle.

For $N = 1$ the formula (18) reduces to the case when all quantities are determined only at the end points of the path. The obtained relations, which are valid for an arbitrary three-dimensionally nonuniform atmosphere, extend the result of Refs. 9 and 10, which was obtained based on simplified atmospheric models.

To study the possible accuracy of the proposed method for determining $\bar{n}$ we performed a numerical experiment, in which real profiles $n(\vec{r})$, obtained under natural conditions for a path 1 km long, were employed.[11]

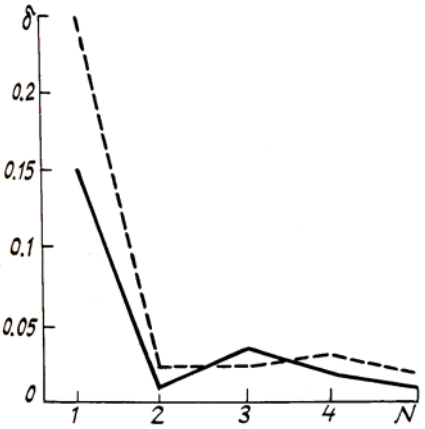

FIG. 1. The error made in determining the index of refraction of air using the formulas (18) (solid line) and (19) (broken line) $|\Delta n| = \delta\cdot 10^{-6}$ as a function of the number of partitions of the region of integration.

To perform the numerical experiment the exact value of $\bar{n}$ was calculated on a computer using available profiles $n(\vec{r})$ and the formula (2); for these profiles $\bar{n}$ was also determined using the formula (18). The difference between these two quantities determines the error $\nabla\bar{n}$ made in determining the index of refraction using the formula (18). Figure 1 shows the absolute magnitude of this difference (solid line) as a function of the number of points on the path at which the local measurements of the index of refraction are performed. The figure also shows the magnitude of the error made in determining n by the traditional trapezoidal method (broken line). One can see that the method under study makes it possible to reduce under given conditions the error in determining the path-averaged index of refraction of air to $1.5\cdot 10^{-7}$ with $N = 1$, and for $N \geq 2$ this error can be made to be much less than 10^{-7}.

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