Simulation of laser beam transformation in a ring interferometer: fractal geometry of a chaotic attractor as applied to information processing

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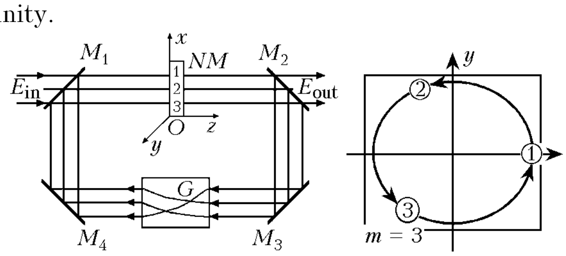
We present a study of the spatially determined chaos in the model of processes occurring in a nonlinear ring interferometer operated in the static mode. Discrete mapping is used as applied to the cases of single- and double-frequency optical radiation. The fractal dimension D_0 serves a quantitative characteristic of the spatially determined chaos. The families of fractal dimension maps of an attractor in the models are constructed based on discrete mapping under various conditions.

In recent years, investigators have actively been searching for principles and the element base of nonlinear-optical tools for information processing. In particular, the possibilities of information protection using the dynamic and spatially deterministic chaos are discussed.[1,2] Thus, the development of facilities for hidden information transfer in the radio region assumes the mode of chaotic oscillations in an encryption device.[3] Chaotic oscillations (modes) usually correspond to *strange* (that is, fractal) attractors.

In solving the problem of information protection and processing in the optical wavelength region, the nonlinear ring interferometer (NRI) is one of the possible prototypes of an encryptor.[2,4] This interferometer is a type of an open ring dynamic system, the optical radiation propagates through. The NRI feedback loop can perform various large-scale transformations of the optical field (focusing of a laser beam, shift and/or turn of the beam in the cross-section plane). Therefore, NRI is capable of generating regular or chaotic spatiotemporal structures in the cross section of a light beam. However, the processes in the NRI model have not still been studied from the viewpoint of estimating the dimension of its attractor as applied to analysis of modes, bifurcations, and NRI suitability for information processing.

Interferometer structure and mathematical model of the processes in the NRI

The optical arrangement of an NRI is shown in Fig. 1, where E_{in} and E_{out} are the fields at the NRI input and output; NM is the nonlinear medium (for example, liquid crystal) with the length L; G is the element of large-scale transformation of the light field (field expansion, shift, or turn); M_1, M_2, M_3, M_4 are mirrors (M_1 and M_2 are semitransparent mirrors with the intensity reflection coefficient R). For M_3 and M_4 the reflection coefficient is equal to unity.



a b

Fig. 1. Ray path in the NRI as the light field turns by $\Delta = 120°$ in the plane xOy: (*a*) trajectories of rays 1, 2, 3 closing upon three passages around NRI; (*b*) projection of the trajectories of rays 1, 2, 3 onto the xOy plane.

Let the radiation at the NRI input be a sum of two quasimonochromatic fields with the amplitudes $a(\mathbf{r}, t)$, $b(\mathbf{r}, t)$ and the frequencies $\omega \pm \Omega$ having circular polarization of different (at $\omega > \Omega$) or identical (at $\omega < \Omega$) signs:

$$E_x(\mathbf{r},t) = a(\mathbf{r},t)\cos[(\omega+\Omega)t + \varphi(\mathbf{r},t) + \psi(\mathbf{r},t)] + b(\mathbf{r},t)\cos[(\omega-\Omega)t + \varphi(\mathbf{r},t) - \psi(\mathbf{r},t)],$$

$$E_y(\mathbf{r},t) = a(\mathbf{r},t)\sin[(\omega+\Omega)t + \varphi(\mathbf{r},t) + \psi(\mathbf{r},t)] - b(\mathbf{r},t)\sin[(\omega-\Omega)t + \varphi(\mathbf{r},t) - \psi(\mathbf{r},t)],$$

where ω (or Ω at $\omega < \Omega$) has the meaning of the mean frequency, and 2Ω (2ω at $\omega < \Omega$) is the frequency difference between the field components. To reflect the specifics of the considered optical field, we operate with the parameter of bichromaticity $q \equiv \Omega/\omega$ (Ref. 5).

Neglecting the diffusion of molecules in the nonlinear medium, from the model proposed in Ref. 5, we can obtain the description of the dynamics of the nonlinear phase progression U in the NRI nonlinear medium in the point approximation. The term "point approximation" means that, depending on the type of the large-scale transformation of the field by the element G in the feedback loop, the whole set of points belonging to the cross section of

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the laser beam in the NRI is divided into the infinite number of mutually independent (in the sense of the absence of physical interaction between the fields and nonlinear phase progression) subsets. But these subsets are essentially the chains of points, at which the light fields and the nonlinear phase progression successively interact (Fig. 1*b*).

In other words, the ray, propagating through the nonlinear medium and the feedback loop of NRI, at the point i (for example, $i = 1, 2, 3$ in Fig. 1*b*) acquires the phase progression U_i and experiences the delay t_{ei}. Due to the element G, the ray comes at the point i+1. Here, in the sum with one of the NRI input rays, it, according to the model from Ref. 5, affects the rate of variation of the nonlinear phase progression U_{i+1}. Note that since the ray paths in Fig. 1 are closed, the index $i + 1 = 4$ must be equal to the index $i + 1 = 1$. Thus, the phase progression U_i at the point i affects the phase progression U_{i+1} at the point $i + 1$. Finally, the ray trajectory becomes closed upon m passages around NRI. From here on, according to the accepted method of enumeration, $i + 1$ denotes the operation $[(i + 1) \bmod m] + 1$, where $(i + 1) \bmod m$ designates the residue of division of $i + 1$ by m. Physically, this means that the ray from the mth point comes to the first one.

Thus, in the point approximation, from the general equations[5] we obtain the system of ordinary differential equations (ODEs):

$$\tau_{ni}\mathrm{d}U_i(t)/\mathrm{d}t = -U_i(t) + f_i,$$

$$f_i \equiv f_i(t) = Kab_{i,i}(t) + pKab_{i-1,i}(t-\tau) + [\gamma_{i-1}(t)/\sigma]\times$$
$$\times\{Ka_{i,i-1}(t,t-\tau)\cos[(1+q)\,\omega\tau + \varphi_i(t) - \varphi_{i-1}(t-\tau) +$$
$$+\,\psi_i(t) - \psi_{i-1}(t-\tau)] + Kb_{i,i-1}(t,t-\tau)\times$$
$$\times\cos[(1-q)\,\omega\tau + \varphi_i(t) - \varphi_{i-1}(t-\tau) -$$
$$-\,\psi_i(t) + \psi_{i-1}(t-\tau)]\}, \qquad (1)$$

where $\tau \equiv \tau_{i-1}(t) = t_{e\,i-1}(t) + U_{i-1}(t - t_{e\,i-1}(t))/\omega$; $\gamma_{i-1}(t)$ is the doubled coefficient of radiation loss for a single trip round the NRI; $p = 0$ in the approximation of high loss, but $p = [\gamma_{i-1}(t)/\sigma/2]^2$ in the approximation of a single passage; "mixed" (Kab) and "partial" (Ka, Kb) nonlinearity parameters are

$$Kab_{i,j}(t) \equiv (1-R)\,n_{2j}lk\,[a_i^2(t) + b_i^2(t)],$$
$$Ka_{i,i-1}(t,t-\tau) \equiv (1-R)\,n_{2i}lk\,a_i(t)\,a_{i-1}(t-\tau),$$
$$Kb_{i,i-1}(t,t-\tau) \equiv (1-R)\,n_{2i}lk\,b_i(t)\,b_{i-1}(t-\tau),$$
$$k = \omega/c,$$

where l is the length of the nonlinear medium; n is the refractive index; σ is the coefficient of expansion (compression) of the light beam.

In the static mode of NRI operation, that is, with the phase progression not changing in time ($\mathrm{d}U/\mathrm{d}t = 0$), the model (1) can be reduced to the recursion. Assume that the static mode occurs at the parameters a_i, b_i, φ_i, ψ_i, γ_i, t_{ei} constant in time. Then from Eq. (1) we have

$$U_i = Kab_{i,i} + pKab_{i-1,i} + [\gamma_{i-1}/\sigma]\times$$
$$\times\{Ka_{i,i-1}\cos[(1+q)(\Phi_{i-1} + U_{i-1}) + \varphi_i - \varphi_{i-1} + \psi_i - \psi_{i-1}] +$$
$$+\,Kb_{i,i-1}\cos[(1-q)(\Phi_{i-1} + U_{i-1}) + \varphi_i - \varphi_{i-1} - \psi_i + \psi_{i-1}]\},$$

where the phase delay in the feedback loop is

$$\Phi_i \equiv \omega t_{ei},\ Kab_{i,j} \equiv (1-R)\,n_{2j}lk\,[a_i^2 + b_i^2],\ Ka_{i,i-1} \equiv$$
$$\equiv (1-R)\,n_{2i}lk\,a_i a_{i-1},\ Kb_{i,i-1} \equiv (1-R)\,n_{2i}lk\,b_i b_{i-1}.$$

According to Ref. 6 (pp. 15–20), the previous equation is a one-dimensional discrete map (DM).

In the case of homogeneity of the optical properties of the NRI nonlinear medium ($n_2 = n_{2j}$) and the amplitudes of the input field ($a = a_i$, $b = b_i$), the equality $Kab = Ka + Kb$ is fulfilled. It is convenient to introduce the sum parameter of nonlinearity K and the fraction Q_a of the power of the component with the frequency $(1 + q)\omega$ by the rule: $K \equiv Kab = (Ka + Kb)$, $Q_a \equiv Ka/K$, then $Ka = K\,Q_a$, $Kb = K(1 - Q_a)$. In the case of homogeneity of the other NRI optical properties ($\Phi = \Phi_i$, $\gamma = \gamma_i$) and the input field ($\psi_i = 0$, $\varphi_i = 0$), we can readily obtain DM for the case of the double-frequency radiation:

$$U_{i+1} = K\,\{1 + p + \gamma\{Q_a\cos[(1+q)\,(\Phi + U_i)] +$$
$$+\,(1 - Q_a)\cos[(1-q)\,(\Phi + U_i)]\}/\sigma\}, \qquad (2)$$

where the value $p = 0$ corresponds to the high loss approximation; and $p = (\gamma/\sigma/2)^2$ corresponds to the approximation of a single passage.

In the case of a single-frequency radiation at the NRI input ($q = 0$) and at $p = 0$, $\sigma = 1$, from Eq. (2) we obtain

$$U_{i+1} = K\,[1 + \gamma\cos(U_i + \Phi)]. \qquad (3)$$

Consider the peculiarities of the NRI operated in the static mode. The interest in this mode is caused by the fact that the possibility of encoding and, correspondingly, decoding of a two-dimensional image has been earlier demonstrated for the NRI in the static mode.[4] This mode is preferable, when the bandwidth of an optical communication channel is a limiting factor or when storage of decoded information is needed. The degree of signal hiding in this case depends on the characteristics of the static mode, which, in its turn, is determined by the combination of the parameters of light field, nonlinear medium, and NRI.

The above possibilities are connected with the fact that because of the system nonlinearity the static mode is capable of providing for randomizing of the spatial distributions of the amplitude and phase of the optical field and the NM refractive index in the xOy plane. This phenomenon discovered when simulating the NRI processes was called *the spatial deterministic chaos* (SDC).[4] It is essentially the static deterministic, but disordered spatial distribution of the amplitude, phase, and NM refractive index.

As known, drawing distributions of some characteristic having a complex dynamics on the plane of parameters, that is, the so-called maps, is an efficient way to study the properties of dynamic systems.[6] One of the significant characteristics of the static mode in the NRI model is the fractal dimension of the attractor corresponding to the SDC phenomenon.

Fractal dimension maps of the attractor in the model based on discrete mapping

Drawing fractal dimension maps in a mathematical model can be helpful in solving the problem on optimizing the parameters and/or modes ensuring the highest degree of hiding for a message masked by a SDC.

To draw maps, the dimensions D of the attractor in the model (3) were calculated, in particular, the Hausdorff–Besicovitch dimension:

$$D_0 = -\lim_{r\to 0}\frac{\ln M(r)}{\ln r},$$

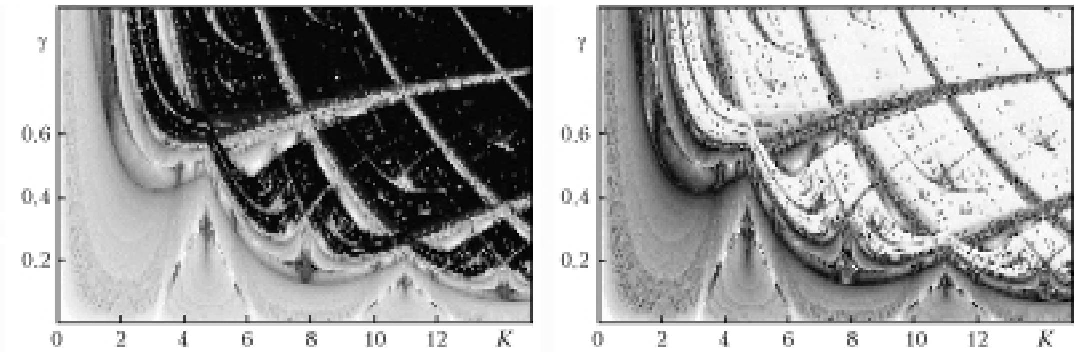
where r is the box size; $M(r)$ is the number of boxes needed to cover the attractor in the phase space.[6]

Figures 2 to 4 depict the maps of dimension $D_0(K, \gamma)$, $D_0(K, \Phi)$, and $D_0(\gamma, \Phi)$ under the initial condition $U_1 = K$. The dark areas in the maps correspond to the maximum of: D_0 (*a*), and the absolute value of D_0 deviation from the nearest integer number, that is, 0 or 1 (*b*). The procedure of verifying the algorithms for drawing these maps for the case of monochromatic radiation, that is, model (3), is described in Ref. 7.

The coincident structures of the fractal dimension maps (Figs. 2*a* and 3*a*) and the maps of the Lyapunov characteristic exponent (LCE) for the same discrete map drawn in the same coordinates as in Ref. 8 (Figs. 4*a* and *b*) can serve an additional verification.

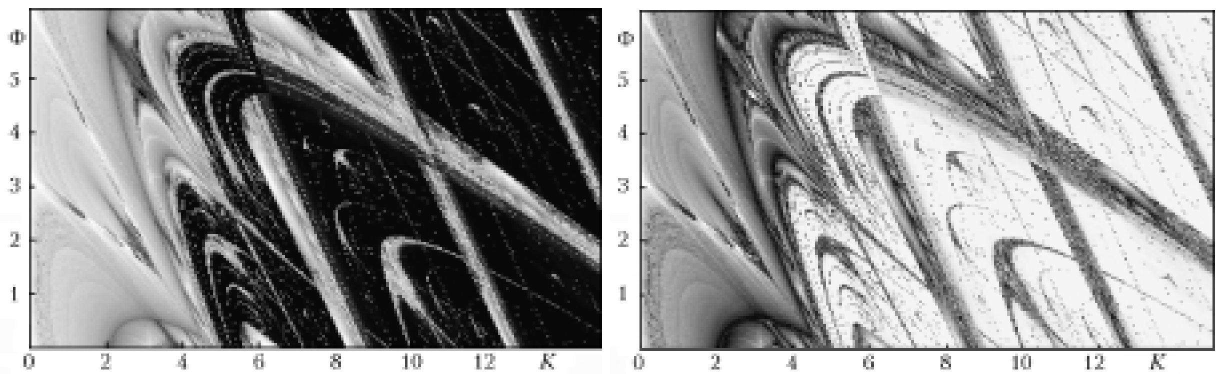
This coincidence of the structures has not only verification, but also the information meaning. It is in the following: it turned out that the basic Kaplan–Yorke hypothesis is true for the considered discrete map (3) modeling the NRI processes. According to this hypothesis, it is possible to determine the fractal dimension D_0 from the spectrum of LCE values. The hypothesis has been proved rigorously only for chaotic attractors of two-dimensional reversible maps [see Ref. 6, p. 190]. But the map (3) is one-dimensional, so the validity of the Kaplan–Yorke hypothesis for it is not *a priori* obvious.

To check the correctness of drawing the maps of the attractor dimension D_0 in the model (2), we took the limiting case of the zero detuning between the components of the spectrum, that is, $q = 0$. The map drawn (Fig. 5) has the same structure as the map corresponding to the monochromatic radiation [model (3)], which is shown in Fig. 2.



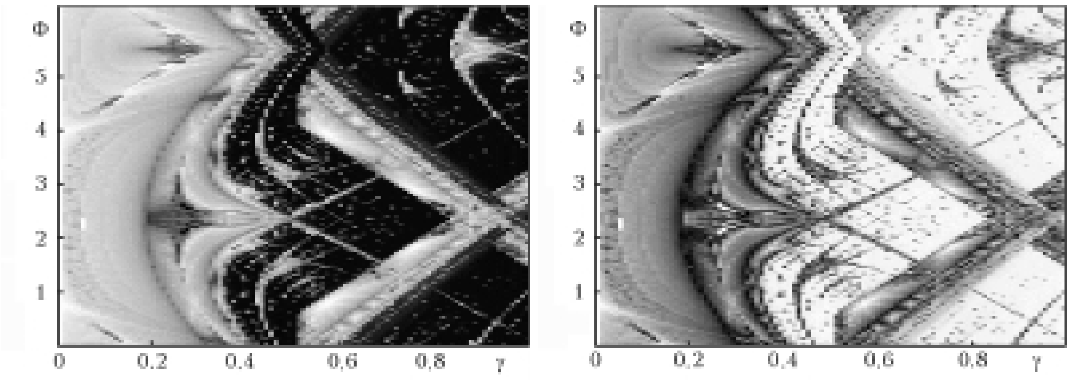
a *b*

Fig. 2. Fractal dimension $D_0(K, \gamma)$ maps of attractor in model (3).



a *b*

Fig. 3. Fractal dimension $D_0(K, \Phi)$ maps of attractor in model (3).



a *b*

Fig. 4. Fractal dimension $D_0(\gamma, \Phi)$ maps of attractor in model (3).

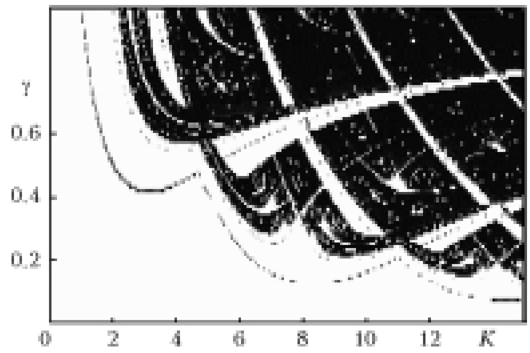


Fig. 5. Map of fractal dimension D_0 of the attractor in model (2) in the limiting case of zero frequency detuning $q = 0$.

Figures 6 and 7 demonstrate the effect of frequency detuning q between the spectral components and the power fraction Q_a of the high-frequency component on the structure of the fractal dimension D_0 maps. At a rather small frequency detuning between the spectral components $q = 0.01$ (Fig. 6) the D_0 map for the double-frequency radiation becomes identical to the case of $q = 0$ (see Fig. 5), which corresponds to the single-frequency radiation.

Judging from the tendency demonstrated in Fig. 6, the structure of the D_0 map changes markedly starting from the frequency detuning $q = 0.05$.

Interpreting the results calculated for the given frequency detuning ($q = 0.1$), one should pay attention to the fact that Fig. 7*b* corresponds to the equal power contributions of the spectral components to the nonlinear effects. The difference between these contributions significantly transforms the structure of the fractal dimension map (Figs. 7*a* and *c*) and makes it more similar to that for the single-frequency radiation (see Fig. 5). Nevertheless, the effect of the contributions of the low- and high-frequency spectral components on the structure of the D_0 maps is asymmetric. Physically, this can be explained by the fact that the nonlinear phase progression U differently depends on the radiation frequency, as can be seen from Eq. (1). This difference, in its turn, manifests itself in the character of nonlinear dynamics.

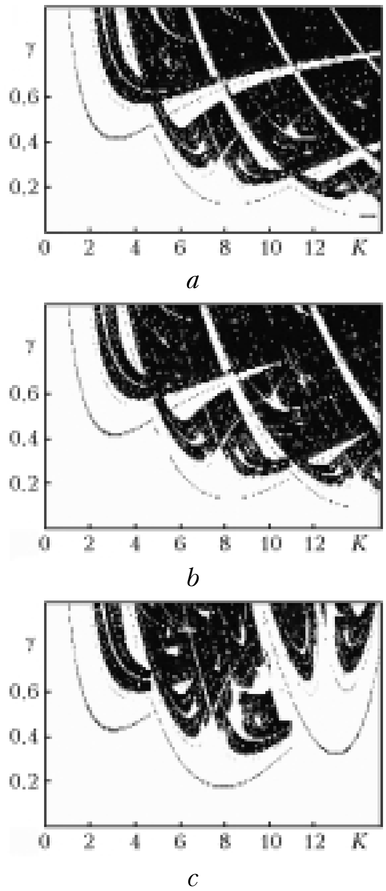
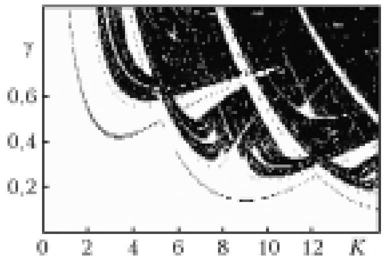


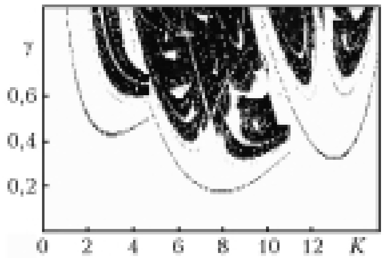
Fig. 6. Structure of the dimension D_0 maps versus the frequency detuning q between the radiation components: $q = 0.01$ (*a*), 0.05 (*b*), and 0.1 (*c*). The components have the same power $Q_a = 0.5$.

Comparison of the structures of the D_0 maps (see Figs. 6 and 7) and LCE maps [Ref. 8, Fig. 10] indicate toward some similarity between them. This stimulates an independent investigation to check the applicability of the Kaplan–Yorke hypothesis in the case of discrete mapping (2) corresponding to the double-frequency radiation.

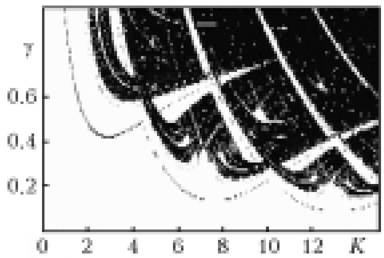
In the case of a single-frequency radiation [model (3)], the experience of simultaneous drawing and studying the fractal dimension maps and bifurcation lines indicates the productivity of interpretation of the map and line structures.[7]



a



b



c

Fig. 7. Structure of the dimension D_0 map versus the power fraction Q_a of the high-frequency spectral component: $Q_a = 0.1$ (*a*), 0.5 (*b*), and 0.9 (*c*); frequency detuning $q = 0.1$.

Therefore, it is quite correct to expect that for the bichromatic radiation the maps can also be interpreted based on drawing the bifurcation lines for the model (2). But this problem is of independent significance and is beyond the scope of this paper.

Conclusions

The models of processes in the NRI have been considered in the high loss approximation. These models use the DM apparatus as applied to the cases of single- and double-frequency radiation at the NRI input. To determine the ranges of parameters corresponding to strange or ordinary attractors in the models, it has been proposed to draw the maps and analyze the dependence of the fractal dimension on the NRI parameters.

The Hausdorff–Besicovitch dimension D_0 has been selected as a quantitative characteristic of the SDC phenomenon occurring in the static NRI mode. The families of D_0 maps have been drawn on the following planes: nonlinearity parameter K–doubled radiation loss coefficient γ; nonlinearity parameter–phase delay in the feedback loop Φ; $\gamma - \Phi$. It has been shown that the structure of the D_0 maps depends on the frequency detuning q between the components of the bichromatic radiation and on the power fraction Q_a of the high-frequency spectral component (Figs. 6 and 7). The plans for the future have been formulated as to draw bifurcation lines for the double-frequency case as a basis for the interpretation of the structure of the corresponding D_0 maps.

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