Optical transfer function of a randomly inhomogeneous dissipative medium with lens properties

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This paper considers the image transfer through a randomly inhomogeneous absorbing medium with lens properties. The optical transfer function of the medium is studied taking into account the effect of the initial beam coherence, complex dielectric constant fluctuations and lens properties of the medium, as well as the role of a receiver. An analytical equation is derived for the optical transfer function under the assumption that a symmetrical refraction channel with fluctuating complex dielectric constant is formed in the medium.

As known, the atmosphere is one of the main information transfer channels. In the problems of image transfer by laser beams, it is usually believed that the atmosphere is characterized, on the average, by a homogeneous random field of the real part of the dielectric constant (refractive index). However, under certain conditions (long paths, high intensity of radiation that bears the information) both the mean spatial inhomogeneity of the image transfer channel (its lens properties) and fluctuations of the imaginary part of the dielectric constant should be taken into account.

This paper considers transfer of an image through a randomly inhomogeneous absorbing medium with lens properties. Such a medium may occur at propagation of a high-intensity laser beam through the atmosphere, in particular, through water-droplet aerosol, in which heating and distortion of the condensed phase by the beam give rise to fluctuations in both the imaginary and real part of the medium dielectric constant, as well as the refraction channel (see, e.g., Refs. 1 and 2).

Let an observed object with the initial distribution of the radiation field $u_{\text{ob}}(\rho)$ be located in the plane $z = 0$ in the refraction channel, whose field of fluctuations of the complex dielectric constant and the lens properties are specified as:

$$\varepsilon(\rho, z) = \varepsilon_R(\rho, z) + i\varepsilon_I(0, z) + (\varepsilon'_R + \varepsilon'_I), \qquad (1)$$

where ε_R is the regular part of ε describing the lens properties of the medium; ε_I is the mean value of the imaginary part of ε; ε' is the random component of ε; z is the coordinate measured along the axis of the refraction channel (in the propagation direction of the radiation forming this channel); ρ is the radius vector in the plane normal to the z axis.

Assume that an optical receiver (presented by an equivalent lens for convenience) is located in the plane $z = z_{\text{r}}$. The effect of the receiver on the radiation that bears the image information is described by the function $W(\rho)\exp(-ik\rho^2/2f)$ with the aperture distribution W and the focal length of the equivalent lens f (k is wave number).

Assuming that the object's image with the distribution of the complex amplitude $u_{\text{im.a}}(z_{\text{im}}, \rho)$ is formed in the plane $z = z_{\text{im}}$, according to the Huygens–Kirchhoff equation, we have

$$u_{\text{im.a}}(z_{\text{im}}, \rho) = \int d^2\rho_1\, u(z_{\text{r}}, \rho_1)\, W(\rho_1)\, e^{ik\rho_1^2/2f}\, G_0(z_{\text{im}}, \rho; z_{\text{r}}, \rho_1), \qquad (2)$$

where $G_0(z_{\text{im}}, \rho; z_{\text{r}}, \rho_1)$ is the Green's function of the free space; $u(z_{\text{r}}, \rho)$ is the complex amplitude of object's radiation once it propagated the distance z_{r} in the refraction channel.

Using Eq. (2) for the image intensity averaged over the ensemble of realizations, we obtain:

$$\langle I_{\text{im.a}}(z_{\text{im}}, \rho)\rangle = \iint d^2\rho_1\, d^2\rho_2\, W(\rho_1)\, W(\rho_2) \times \exp[-ik(\rho_1^2 - \rho_2^2)/2f] \times G_0(z_{\text{im}}, \rho; z_{\text{r}}, \rho_1)\, G_0^*(z_{\text{im}}, \rho; z_{\text{r}}, \rho_2)\, \Gamma_2(z_{\text{r}}, \rho_1, \rho_2), \qquad (3)$$

where $\Gamma_2(z_{\text{r}}, \rho_1, \rho_2) = \langle u(z_{\text{r}}, \rho_1) u^*(z_{\text{r}}, \rho_2)\rangle$ is the function of wave coherence in the channel.

Since it is commonly believed that in the interesting cases of an extended object, which is located far from the receiver ($z_{\text{r}} \gg z_{\text{im}} - z_{\text{r}}, f$) the best practice is to form the image at the focus of the receiving system. In this case the following is valid: $z_{\text{im}} - z_{\text{r}} \approx f$. Taking all the above-said into account and using the explicit form of the Green's function

$$G_0(z_{\text{im}}, \rho; z_{\text{r}}, \rho_1) = \frac{k}{2\pi i(z - z_1)} \times \exp[-ik(\rho - \rho_1)^2/2(z - z_1)], \qquad (4)$$

as well as changing the variables $R = (\rho_1 + \rho_2)/2$, $r = \rho_1 - \rho_2$, we can transform Eq. (3) to a more compact form

$$\langle I_{\text{im.a}}(z_{\text{im}}, \rho)\rangle = (k^2/4\pi^2 f^2) \iint d^2R\, d^2r \times \exp(-ik\rho\mathbf{r}/2f)\, W(\mathbf{R} + \mathbf{r}/2)\, W(\mathbf{R} - \mathbf{r}/2)\, \Gamma_2(z_{\text{r}}, \mathbf{R}, \mathbf{r}). \qquad (5)$$

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According to Ref. 5, the optical transfer function (OTF) of a "random medium – receiver" system is related to the mean intensity as

$$<I_{\text{im.a}}(\rho)> = (k^2/4\pi^2 f^2) \int d^2 r <T_{\text{t.f}}(\mathbf{r})> \times$$
$$\times \exp(-ik\rho\mathbf{r}/2f). \qquad (6)$$

Comparing this equation with Eq. (5) for OTF, we obtain:

$$<T_{\text{t.f}}(\mathbf{r})> = \int d^2 R W(\mathbf{R} + \mathbf{r}/2)\, W(\mathbf{R} - \mathbf{r}/2) \times$$
$$\times \Gamma_2(z_{\text{r}}, \mathbf{R}, \mathbf{r}). \qquad (7)$$

It follows from Eq. (7) that the optical transfer function is determined by the aperture of the receiving system and by the function of radiation coherence in the refraction channel.

Let us perform further calculations for the modeled, by a symmetric refraction, channel with complex dielectric constant fluctuations. In this case, the coherence function Γ_2 has the following form[3]:

$$\Gamma_2(z;\, \mathbf{R}, \mathbf{r}) = \int d^2\kappa e^{i\kappa\mathbf{R}} F(z, \kappa, \mathbf{r}), \qquad (8)$$

where

$$F(z, \kappa, \mathbf{r}) = F_0[0, \kappa_0(\kappa, \mathbf{r}, z), \mathbf{r}_0(\kappa, \mathbf{r}, z)] \times$$
$$\times \exp\left\{k^2 \int\limits_0^z d\xi A_{II}(\xi, 0) - k \int\limits_0^z d\xi \varepsilon_I(0, \xi) - \right.$$
$$\left. \frac{k^2}{4} \int\limits_0^z d\xi \sum_{q=R,I} D_{qq}[\mathbf{r}h_1(\xi, z) + \kappa h_2(\xi, z)/k]\right\}, \qquad (9)$$

$$F_0(0, \kappa, \mathbf{r}) = (2\pi)^{-2} \int d^2 R e^{i\kappa\mathbf{R}} \times$$
$$\times <u_{\text{ob}}(\mathbf{R} + \mathbf{r}/2)\, u^*_{\text{ob}}(\mathbf{R} - \mathbf{r}/2)>;$$
$$\mathbf{r}_0(\kappa, \mathbf{r}, z) = \mathbf{r}h_2'(z, 0) - \kappa h_2(z, 0)/k;$$
$$\kappa_0(\kappa, \mathbf{r}, z) = \kappa h_1(z, 0) - k\mathbf{r}h_1'(z, 0),$$
$$h_j'(z, \xi) = dh_j(z, \xi)/dz, \quad j = 1, 2;$$
$$D_{qq}(z, \rho) = A_{qq}(z, 0) - A_{qq}(z, \rho)$$

are structure functions of δ-correlated fluctuations of the real ($q = R$) and imaginary ($q = I$) components of ε;

$$A_{qq}(z, \rho) = 2\pi \int d^2\kappa\, \Phi_{qq}(z, \kappa) \cos(\kappa\rho);$$

$\Phi_{qq}(z, \kappa)$ is the spectrum of fluctuations of the real ($q = R$) and imaginary ($q = I$) parts of ε.

The functions h_j are solutions of the characteristic equation

$$d^2 h_j(z, \xi)/dz^2 \pm \beta^2(z) h_j(z, \xi) = 0$$

(minus sign for the defocusing channel and plus sign for the focusing channel) with the boundary conditions

$$h_1(z, \xi) = h_2'(z, \xi)\big|_{z=\xi} = 1,\ h_2(z, \xi) = h_1'(z, \xi)\big|_{z=\xi} = 0,$$

β is a refraction parameter.

Equation (7) obtained for OTF with the allowance made for Eqs. (8) and (9) can be analyzed numerically or analytically for various forms of the aperture function W, the amplitude distribution of the object's radiation, and characteristic of the refraction channel.

Consider the situation that the object's radiation has the Gaussian amplitude distribution:

$$u_{\text{ob}}(\rho) = A_0 \exp(-\rho^2/2a_0^2).$$

In this case, the function $F_0(0, \kappa, \mathbf{r})$ takes the following form:

$$F_0(0, \kappa, \mathbf{r}) = (A_0^2 a_0^2/4\pi) \exp\{-r^2/4a_0^2 - a_0^2\kappa^2/4\}, \quad (10)$$

where a_0 is the effective size of the object.

Substituting Eqs. (8) and (9) [with the allowance for Eq. (10)] into Eq. (7) and specifying the aperture function W in the form $W(\rho) = \exp(-\rho^2/2a_{\text{r}}^2)$ (where a_{r} is the receiver's aperture radius), we obtain for the OTF of the "refraction channel – receiver" system the following expression:

$$<T_{\text{t.f}}(\mathbf{r})> =$$
$$= \frac{A_0^2 a_0^2 a_{\text{r}}^2}{4} \exp\left\{k^2 \int\limits_0^{z_{\text{r}}} d\xi A_{II}(\xi, 0) - k \int\limits_0^{z_{\text{r}}} d\xi \varepsilon_I(0, \xi)\right\} \times$$
$$\times \exp[-r^2 p(z_{\text{r}})/4a_{\text{r}}^2] \int d^2\kappa \exp\left\{-\kappa^2 r_{\text{e}}^2/4 + krg(z_{\text{r}}) - \right.$$
$$\left. - \frac{k^2}{4} \int\limits_0^{z_{\text{r}}} d\xi \sum_{q=R,I} D_{qq}[\mathbf{r}h_1(\xi, z_{\text{r}}) + \kappa h_2(\xi, z_{\text{r}})]/k\right\}. \quad (11)$$

Here

$$p(z) = 1 + a_{\text{r}}^2 f(z)/a_0^2;\ f(z) = h_2'^2(z, 0) + k^2 a_0^4 h_1'^2(z, 0);$$
$$g(z) = k a_0^2 h_1(z, 0) h_1'(z, 0) + h_2(z, 0) h_2'(z, 0)/k a_0^2;$$
$$r_{\text{e}}^2(z) = a_{\text{r}}^2 + r_{\text{p}}^2(z);\ r_{\text{p}}^2(z) = a_0^2[h_1(z, 0) + h_2(z, 0)/k a_0^2]$$

is the refraction change in the size of the beam coming from the object after it traveled the distance z in the refraction channel.

For further analysis of Eq. (11), we have to define the structure function D_{qq} of the dielectric constant fluctuations in the channel. For this purpose, let us use the square approximation of D_{qq} that is usually used in the problems of wave propagation through the turbulent atmosphere with the Kolmogorov spectrum of fluctuations ε (see, e.g., Ref. 4). Omitting the intermediate transformations, we obtain the following formula for the OTF of the "refraction channel – receiver" system:

$$<T_{\text{t.f}}(r)> = [\pi A_0^2 a_0^2 a_{\text{r}}^2/(a_{\text{r}}^2 + r_{\text{p}}^2 + r_{\text{f}}^2)] \times$$
$$\times \exp\left\{-k \int\limits_0^{z_{\text{r}}} d\xi \varepsilon_I(0, \xi) + k^2 \int\limits_0^{z_{\text{r}}} d\xi A_{II}(\xi, 0)\right\} \times$$
$$\times \exp\left\{-\frac{r^2}{4a_{\text{r}}^2}[1 + a_{\text{r}}^2 f(z_{\text{r}})/a_0^2 + 4a_{\text{r}}^2/r_{\text{c}}^2 - \right.$$

$$-\frac{a_r^2[g(z_r)-4c(z_r)/kr_c^2]^2}{a_r^2+r_p^2+r_f^2}\Bigg\},\qquad(12)$$

where $r_f^2 = 4b(z_r)/kr_c^2$ is the fluctuation change in the size of the beam coming from the object at the distance z_r;

$$r_c^2 = \left[-0.41k^2(C_{RR}^2 + C_{II}^2)l_0^{-1/3}\int\limits_0^{z_r} d\xi h_1^2(\xi,\, z_r)\right]^{-1};$$

$$b(z_r) = \int\limits_0^{z_r} d\xi h_2^2(\xi,\, z_r)\Big/\int\limits_0^{z_r} d\xi h_1^2(\xi,\, z_r);$$

$$c(z_r) = \int\limits_0^{z_r} d\xi h_1(\xi,\, z_r)h_2(\xi,\, z_r)\Big/\int\limits_0^{z_r} d\xi h_1^2(\xi,\, z_r);$$

C_{qq}^2 are structure characteristics of fluctuations of ε (real ($q = R$) and imaginary ($q = I$) parts).

From the obtained equation (12) it can be seen that the optical transfer function in the considered case is determined by a complicated combination of the effects of refraction and fluctuation broadening and loss of coherence in the channel, as well as it depends on the size of the object and the receiver's aperture.

Consider a more detailed situation that diffraction of radiation on the path from the object to the receiver is insignificant (approximation of a plane unlimited wave) and the object emits partially coherent radiation with the initial coherence length ρ_{c_0}. In this case, the function $F_0(0,\, \kappa,\, \mathbf{r})$ has the form

$$F_0(0,\, \kappa,\, \mathbf{r}) = A_0^2\delta(\kappa)\exp(-r^2/4\rho_{c_0}^2),\qquad(13)$$

and the equation for OTF transforms as follows:

$$\langle T_{\text{t.f}}(r)\rangle = [\pi A_0^2a_r^2/h_1^2(z_r,\, 0)]\times$$

$$\times\exp\left\{-k\int\limits_0^{z_r} d\xi\varepsilon_I(0,\, \xi) + k^2\int\limits_0^{z_r} d\xi A_{II}(\xi,\, 0)\right\}\times$$

$$\times\exp\left\{-\frac{r^2}{4a_r^2}\Big[1 + k^2a_r^4h_1'^2(z_r,\, 0)/h_1^2(z_r,\, 0)\Big] -\right.$$

$$\left.-\frac{r^2}{4\rho_{c_0}^2h_1^2(z_r,\, 0)} - \frac{r^2}{4r_{c_1}^2(z_r)}\right\},\qquad(14)$$

where r_{c_1} is the change of the coherence length due to interaction of the radiation with random inhomogeneities of the complex dielectric constant of the refraction channel:

$$r_{c_1}^2 =$$

$$= \left[-0.41k^2(C_{RR}^2 + C_{II}^2)l_0^{-1/3}\int\limits_0^{z_r} d\xi h_1^2(\xi,\, z_r)/h_1^2(z_r,\, 0)\right]^{-1}$$

at $\kappa_0 r(\xi) < 1$,

$$r_{c_1}^2 = \left[-0.37k^2(C_{RR}^2 + C_{II}^2)l_0^{-1/3}\times\right.$$

$$\left.\times\int\limits_0^{z_r} d\xi h_1^{5/3}(\xi,\, 0)/h_1^{5/3}(z_r,\, 0)\right]^{-6/5}$$

at $\kappa_0 r(\xi) > 1$,

$r(\xi) = rh_1(\xi,\, 0)/h_1(z_r,\, 0)$, $\kappa_0 = 5.92/l_0$, l_0 is the inner scale of turbulence.

Compare now Eq. (14) with the optical transfer function $\langle T_{\text{t.f}}(r)\rangle_0$ for a nonabsorbing ($\varepsilon_I = 0$, $C_{II}^2 = 0$) turbulent medium having no lens properties ($\beta \to 0$):

$$\langle T_{\text{t.f}}(r)\rangle_0 =$$

$$= \pi A_0^2a_r^2\exp\Big\{-r^2/4a_r^2 - r^2/4\rho_{c_0}^2 - r^2/4r_{c_0}^2(z_r)\Big\},\qquad(15)$$

where

$$r_{c_0}^2 = [0.41k^2C_{RR}^2l_0^{-1/3}z_r]\ \text{at}\ \kappa_0 r < 1;$$

$$r_{c_0}^2 = [0.37k^2C_{RR}^2z_r]\ \text{at}\ \kappa_0 r > 1.$$

As can be seen from Eqs. (14) and (15), the presence of regular refraction in the channel leads, on the one hand, to variation of the initial coherence length [factor $h_1^2(z_r)$ of $\rho_{c_0}^2$ in Eq. (14)], as well as of the degree of loss of coherence due to wave interaction with random inhomogeneities of ε (the functions $h_1(\xi,\, 0)$ and $h_1(z_r,\, 0)$ entering into $r_{c_1}^2$) and, on the other hand, to some modification of the effect of the receiver's aperture on radiation due to its defocusing (focusing) in the refraction channel [the term $k^2a_r^4h_1'^2(z_r)/h_1^2(z_r)$ in the square brackets at $r^2/4a_r^2$ in Eq. (14)]. The contribution of fluctuations of the imaginary part of ε to OTF manifests itself as some effective decrease of the mean extinction and in additional variation of wave coherence (the term C_{II}^2 in the equation for $r_{c_1}^2$).

Let us calculate the degree of spatial resolution of the "refraction channel – receiver" system. The minimum resolvable size δl can be determined through the function F_R characterizing the quality of the joint optical system as follows[5]:

$$\delta l = \frac{1}{2F_R^{1/2}},\ F_R = \int d^2\kappa\,\langle T_{\text{t.f}}(\kappa)\rangle_n,\qquad(16)$$

where

$$\langle T_{\text{t.f}}(\kappa)\rangle_n = \langle T_{\text{t.f}}(\mathbf{r})\rangle/\langle T_{\text{t.f}}(0)\rangle\big|_{\mathbf{r}=\kappa f/k}$$

is the normalized optical transfer function.

From Eq. (16) and taking into account Eq. (14), for δl of the system "refraction channel – receiver" we obtain

$$\delta l = f/(4\sqrt{\pi}ka_x),\qquad(17)$$

where

$$a_x^2 = a_{r_1}^2[1 + a_{r_1}^2/\rho_{c_1}^2 + a_{r_1}^2/r_{c1}^2]^{-1};$$

$$a_{r_1}^2 = a_r^2[1 + k^2 a_r^4 h_1'^2(z_r)/h_1^2(z_r)]^{-1};\ \rho_{c_1}^2 = \rho_{c_0}^2 h_1^2(z_r).$$

Let us analyze Eq. (17) at various parameters of the problem. In the case of propagation of a fully coherent radiation from the object in the medium without random inhomogeneities, δl is determined by the parameter a_{r_1}, which in the longitudinally homogeneous refraction channel takes the form

$$a_{r_1}^2 = a_r^2[1 + k^2 a_r^4/R_n^2]^{-1},$$

where R_n is the effective focal length of the refraction channel. It is obvious that in this medium (unlike the medium without such a channel, where δl continuously decreases with the increasing receiver's aperture), the minimum resolvable distance does not monotonically decrease with increasing a_r, but reaches its minimum at $a_{r_1} = (R_n/k)^{1/2}$ and then increases.

If the object's radiation is partially coherent and the image is transferred through a random medium without a thermal lens, then, because of no restriction imposed on the aperture size in this case, the minimum value of δl is determined by the relation between ρ_{c_0} and $r_{c_0}(z_r)$:

$$\delta l_{\min} = f/4\sqrt{\pi}k\rho_{c_0} \text{ at } r_{c_0}(z_r) > \rho_{c_0},$$

$$\delta l_{\min} = f/4\sqrt{\pi}kr_{c_0} \text{ at } r_{c_0}(z_r) < \rho_{c_0}.$$

It is clear that as the path length in the medium, through which the image is transferred, increases, the loss of coherence begins to dominate, since $r_{c_0}(z_r)$ tends to zero with the increasing z_r. Thus, in an extended turbulent medium the loss of coherence leads to a considerable loss in resolution of the "medium–receiver system."

The situation is somewhat different in random media with thermal lenses. In such media, the minimum value of δl is determined by the relation between three parameters: a_{r_1}, ρ_{c_1}, and r_{c_1}. In the defocusing channel (where $h_1(z_r) = \cosh(\beta z_r)$ the role of the initial incomplete coherence of radiation becomes insignificant, since $\rho_{c_1} = \rho_{c_0}\cosh(\beta z_r)$ increases with increasing z_r. Besides, unlike the regularly homogeneous medium, the wave coherence length r_{c_1} in the defocusing clearing-up channel does not tend to zero with increasing z_r, but saturates at some limiting value $r_{\text{c.lim}} = [0.22k^2(C_{RR}^2 + C_{II}^2)\beta^{-1}]^{-3/5}$. The corresponding value of $\delta l_{\lim} = f/(4\sqrt{\pi}kr_{\text{c.lim}})$ can be much less than in the medium without a thermal lens.

In conclusion it should be noted that the presence of the imaginary part of ε fluctuations somewhat decreases the system resolution as compared to the transparent defocusing random medium, since it leads to a decrease of $r_{\text{c.lim}}$ due to the term C_{II}^2.

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