Optical turbulence modes in a nonlinear optical system with time-delayed distributed feedback

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The chaotic dynamics of light fields formed by a distributed nonlinear optical system with time-delayed feedback and diffraction is studied numerically. In a certain range of system parameters, these fields are found to have isotropic spectra of spatial fluctuations decreasing smoothly with the increasing spatial frequency. The possibility of using these systems as generators of artificial optical turbulence is discussed.

Introduction

The chaotic dynamics of distributed optical systems with the third-order nonlinearity and a feedback has been the subject of intense investigations in recent years.[1–7] Already in the first papers devoted to this subject, it was assumed that such systems could be used to model the so-called optical turbulence under laboratory conditions.[1,2] These assumptions were based on the fact that optical radiation formed by such a system under certain conditions has statistical characteristics similar to the analogous characteristics of radiation having passed through a natural turbulent medium, for example, the atmosphere.

The control over stochastic dynamics of such systems opens up the possibility of changing characteristics of spatial fluctuations of the field. In a particular case, it may be control over the scale of inhomogeneities and, in a more general case, variation of the whole profile of the spectral power density (SPD) of fluctuations. Such a control seems to be rather difficult in field experiments on radiation propagation in natural randomly inhomogeneous media. Thus, the study of statistical properties of chaotic modes in the considered systems is of a significant practical interest.

Experimental studies of the chaotic behavior of systems with field transformation in the feedback loop started in mid-90s. Different types of field transformation were studied: turn,[2,5] diffraction,[3] as well as their combination.[4] A distinctive feature of all these systems is that the transition to chaotic modes at the increasing intensity of the input radiation can be characterized as stochastization of regular structures: reverberators in a system with turn and hexagonal structures in a system with diffraction. Therefore, when the space-time chaotic behavior is observed in a system, there are significant field correlations on the spatial scales corresponding to the scales of these regular structures. This is a serious disadvantage, if we consider such systems as potential generators of artificial optical turbulence.

In this connection, particular interest is paid to systems with the so-called local time instability, an example of which is the distributed system with a delayed feedback. In this system, chaotic field fluctuations are not caused by non-local spatial relations, as in systems with geometric field transformations, or by phase-amplitude Fourier filtering, as in the system with the diffraction, but by spatially local while temporally non-local nonlinear interaction, which does not lead to undesirable spatial correlations of the field.[7]

It should be noted that the delayed system was, in fact, the first system, from which the active study of bistability and chaos in optics started.[8] A number of papers were devoted to the study of transitions from stationary modes to chaotic ones for a one-dimensional system with diffraction (see, for example, Ref. 9 and references therein). The possibility of using delayed systems as chaos generators for enciphering at optical information transfer was considered in Ref. 10. However, as far as we know, no detailed study of statistical characteristics of chaotic modes was conducted. In our opinion, this paper could fill this gap. It presents the results of numerical simulation of a distributed system with a delayed feedback. The results of this simulation are used for calculation of statistical characteristics of chaotic modes in a wide range of control parameters, and the effect of diffraction on the system dynamics is studied. The possibility of applying such systems as generators of optical turbulence is discussed as well.

1. Model

A nonlinear optical system with the Kerr-type nonlinearity and some field transformation in the feedback loop is described by the nonlinear diffusion equation for phase modulation of the light field $u(\mathbf{r},t)$ (Ref. 1):

$$\tau \frac{\partial u(\mathbf{r}, t)}{\partial t} + u(\mathbf{r}, t) = D\Delta u(\mathbf{r}, t) + \tilde{K}\,|A_{FB}(\mathbf{r}, t)|^2. \quad (1)$$

Here τ is the relaxation time; D is the diffusion coefficient; K is the parameter of nonlinearity; A_{FB} is the field amplitude in the feedback loop. This equation can be used for description of the two best-known classes of systems with a feedback: a nonlinear ring cavity in the single-pass approximation[8] and a system based on liquid-crystal (LC) transparency.[1] All the results presented here can be extended to both of the classes, however in what follows we believe that the LC system is a prototype for the considered model (Fig. 1*a*).

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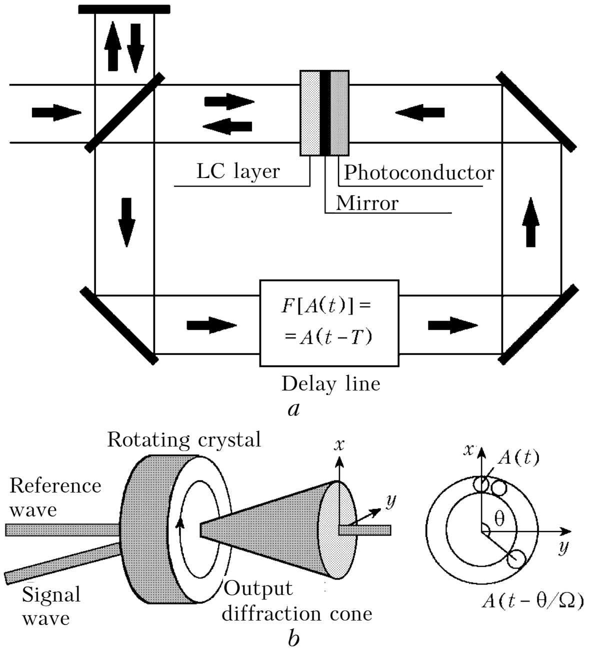

Fig. 1. Distributed optical system with a delay line in the feedback loop; Ω is the angular rotation rate of the crystal.

The equation for the field amplitude A_{FB} should be written with the allowance for a particular form of the transformation in the feedback loop; in our case, it is delay and diffraction. For relatively slow LC systems with the relaxation time $\tau \sim 10^{-2}$–10^{-3} s, the delay of the optical signal can be introduced artificially, for example, using a specialized photorefractive delay line,[11] which is shown schematically in Fig. 1*b*. In accordance with this scheme, two laser beams enable recording a holographic array in a photorefractive crystal. As a result of crystal rotation around the propagation axis of the reference beam, the information about the signal recorded in the previous time arises in the output diffraction cone. To avoid spatial overlap of 2D signals, discrete rotation of the crystal is needed.

However, if time resolution in the delay line Δt is much smaller than the relaxation time of the LC transparency τ, then mathematical description of the delay line with continuous time is suitable. For the delay time T and relaxation time τ such that $T/\tau \approx 5$ (this ratio makes possible the appearance of chaotic modes in a wide range of parameters), as well as for the total number of stored holograms $M \approx 100$, we have $\tau/\Delta t \approx 20$. Since the time of hologram re-recording is much shorter than the relaxation time of the LC transparency, the approximation of continuous time is quite justified.

Taking into account the delay and diffraction in the feedback loop, in the approximation of a plane input wave we have[7]:

$$A_{FB}(\mathbf{r},\ t) = A_0\ \tilde{F}^{-1}[\mathrm{e}^{-i(k_x^2+k_y^2)Z_0}\ \tilde{F}(1 + \gamma \mathrm{e}^{iu(\mathbf{r},\ t-T)+\psi_0})], \quad (2)$$

where A_0 is the amplitude of the input field; T is the delay time; Z_0 is the diffraction parameter (proportional to the diffraction length); γ and ψ_0 are the parameters describing the ratio of interference contributions of the reference signal and the signal reflected from the LC transparency[1]; $\tilde{F}$ and $\tilde{F}^{-1}$ are the direct and inverse Fourier transform operators. The approximation of the plane input wave is valid in most cases, when the spatial scale of field inhomogeneities l (structure scale or correlation length for random fields) is much smaller than the beam aperture Y. Periodic boundary conditions that are also applicable in the case of $l \ll a$ were used for Eq. (1). The initial conditions for phase modulation were chosen in the form $u(\mathbf{r},\ t = 0) = u_0 + \xi(\mathbf{r})$, where u_0 is some constant; $\xi(\mathbf{r})$ describes small noise fluctuations.

There are no general analytical methods for solving Eqs. (1)–(2) under these boundary conditions at arbitrary values of the control parameters K, D, γ, ψ_0, T, and τ. Nevertheless, passing to a spatially discrete model, we can draw some principle conclusions about the system dynamics. Assume, first, that there is no diffraction ($Z_0 = 0$). Replace the function $u(\mathbf{r},\ t)$ with its grid approximation – a set of N^2 values of $u_{ij}(t) = u(i\Delta x,\ j\Delta y,\ t)$. In this case, in place of the equation for phase modulation in the distributed medium (1), we obtain the system of coupled ordinary differential equations for the functions $u_{ij}(t)$:

$$\tau \frac{\partial u_{ij}(t)}{\partial t} + u_{ij}(t) =$$

$$= K\ \{1 + \gamma \cos[u_{ij}(t - T) + \psi_0]\} + \Delta_{ij} u(t), \quad (3)$$

where $K = \tilde{K}|A_0|^2$; $\Delta_{ij}u(t)$ is the linear combination of phase values at neighboring spatial points, whose form depends on the grid approximation of the Laplace operator Δ. Thus, as a result of grid approximation, the distributed medium is replaced by a set of N^2 diffusion-coupled nonlinear oscillators. In the absence of spatial correlations ($D = 0$), we obtain the equation describing the dynamics of an individual oscillator[1]:

$$\tau \frac{\partial u(t)}{\partial t} + u(t) = K\ \{1 + \gamma \cos[u(t - T) + \psi_0]\}. \quad (4)$$

Temporal nonlocality in Eq. (4) does not allow analytical study of the system dynamics at an arbitrary set of parameters. Therefore, following the method proposed in Ref. 8, consider the approximation of the immediate response ($\tau \to 0$). Then, passing on to a discrete time $t_n = nT$ ($n \in Z$) we obtain in place of Eq. (4) with continuous time

$$u^{n+1} = K\ (1 + \gamma \cos[u^n + \psi_0]), \quad (5)$$

where $u^n = u(t_n)$. This equation describes the transition to chaos through period doubling bifurcations at the increasing control parameter K (intensity of input radiation). Such a dynamics of light beams in a ring cavity was studied thoroughly in the 80s.[8,12]

Thus, a distributed system can be approximately considered as a set of diffusion-coupled chaotic oscillators located at the nodes of a 2D grid. For simplicity, assume first that there are no spatial connections between oscillators. Then different zones of the nonlinear medium capable of evolving chaotically remain spatially independent. Since chaos is exponentially unstable motion, infinitely small initial spatial fluctuations are sufficient for the light field to take the spatially chaotic form with time. It is clear that, in the presence of such spatial interactions as diffusion in an actual system, competition arises between two opposite processes, namely, diffusion "smoothing" of spatial inhomogeneities, on the one hand, and their development due to temporal chaos, on the other hand. If in the absence of diffusion the light field in the chaotic mode becomes spatially delta-correlated with time, then in the case of relatively weak diffusion in the system one should expect the appearance of spatiotemporal chaotic modes with nonzero spatial correlation length. In this case, it is rather natural to assume that phase correlations at two spatial points decrease monotonically as the distance between the points increases, and the SPD of spatial fluctuations has a monotonically decreasing character.

Consider now the contribution of diffraction ($Z_0 \neq 0$). In this case, analytical consideration is even more complicated, therefore let us restrict ourselves to the limiting case of a system with a short delay $T \to 0$. As known, diffraction in this case leads to selectivity of the spatial Fourier components of the light field amplitude, due to which both regular and chaotic spatial irregularities can arise in the system.[13] In our case, the equation for linear Lyapunov indices of the spectral phase components $u(\mathbf{r}, t) = \bar{u}(t) + \sum\limits_{\mathbf{k}\neq 0} a_{\mathbf{k}}(t)\, e^{i\mathbf{kr}}$ has the form[14]

$$\lambda_{\mathbf{k}} = -1 - Dk^2 + $$
$$+ 2K\,[\sin k^2 Z_0 + \gamma \sin(k^2 Z_0 + \psi_0 - \bar{u})]. \qquad (6)$$

In the stationary mode, the aperture mean phase $\bar{u}$ is some constant, and according to Eq. (6) the spectral space can be divided into excitation zones, for which $\lambda_{\mathbf{k}} > 0$. Because of the periodic character of the harmonic function, these zones look like concentric rings. The competitive dynamics at low diffusion and relatively small values of the parameter $K \sim 1$ leads to the formation of structures, and at $K \gg 1$ chaotic modes are observed, in which the SPD of spatial phase fluctuations has a nonmonotonic character like damped oscillations.[3] These oscillations are determined just by the presence of concentric excitation zones: the amplitudes of spectral components of fluctuations in excitation zones are far higher than the amplitudes of other components.

Thus, based on this consideration, we can draw the following conclusions about the statistical properties of chaotic modes:

1) at small values of the diffraction parameter Z_0, chaotic modes with the monotonically decreasing SPD of spatial fluctuations can arise in the system;

2) as the parameter Z_0 increases, the effect of diffraction on the system dynamics can lead to appearance of such modes, in which the SPD has a nonmonotonic form.

These conclusions were checked through direct numerical simulation. Calculations were conducted both in the immediate response approximation and with the allowance for relaxation. In the immediate response approximation, Eq. (1) was replaced with the grid analog

$$u^{n+1}(i, j) = D\Delta u^{n+1}(i, j) + K|A_{FB}|^2, \qquad (7)$$

or

$$u_{ij}^{n+1} =$$
$$= \tilde{F}^{-1}\left(\frac{\tilde{F}\{\tilde{K}|A_0|^2|F^{-1}[\tilde{F}(\gamma e^{i[u_{ij}^n+\psi_0]}+1)\,e^{-i(f_i^2+f_j^2)Z_0}]\}}{1+4\pi^2 D(f_i^2+f_j^2)}\right), \qquad (8)$$

where n is the number of the time step.

2. Analysis of statistical characteristics

Figure 2 shows examples of random phase distributions obtained in the chaotic modes for the system without diffraction. It can easily be seen that the characteristic spatial scale of fluctuations (correlation length r_{corr}), as could be expected, increases with the increasing diffusion coefficient. Thus, one could expect that the SPD of phase fluctuations is monotonically decreasing and the SPD width $\Delta f \approx \pi / r_{\text{corr}}$ decreases with the increasing diffusion coefficient D.

The spatial SPD of phase fluctuations $G(f)$ averaged over large number of realizations and over the azimuth angle is shown in Fig. 3*a*. The obtained dependences $G(f)$ actually have the monotonic form and are well approximated by the Gaussian functions $G^{\text{app}}(f) = Ae^{-Bf^2}$ with the approximation error $\chi = \dfrac{\int [G(f) - G^{\text{app}}(f)]^2\,df}{\int G^2(f)\,df} \sim$ $\sim 10^{-3}$. Such a behavior of the SPD of phase fluctuations agrees with the results obtained for grid logistic maps in the mode of developed spatiotemporal chaos.[15]

The SPD width actually decreases with the increasing diffusion coefficient. The coefficient B determining the width of the functions $G_{\text{app}}(f)$ has the dimensionality of L^2. Thus, the dependence $B(D)$ should be linear, that can easily be seen in Fig. 3*b*. This result has a simple physical interpretation: the characteristic scale of spatial inhomogeneities (correlation length) increases with the increasing strength of spatial connections (diffusion coefficient) as $r_{\text{corr}} \sim \sqrt{D}$. Similar linear dependence of the parameters $B \sim D$ was also obtained for chaotic modes in related LC systems with periodic excitation.[6]

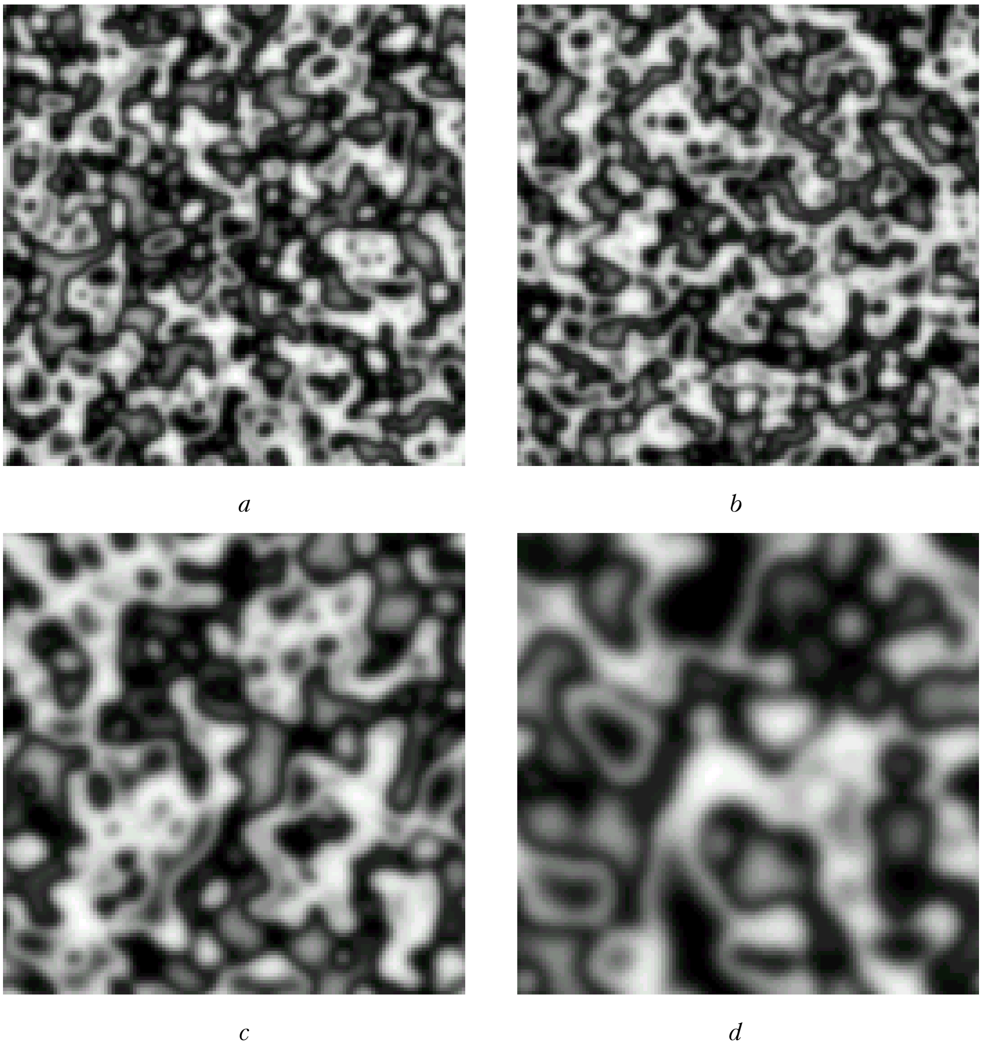

Fig. 2. Spatial phase distributions in chaotic modes (immediate response approximation): $K = 2.65$, $\gamma = 1$, $\psi_0 = 0$, $Z_0 = 0$, $D = 4 \cdot 10^{-5}$ (a, b), $1 \cdot 10^{-4}$ (c), $2 \cdot 10^{-4}$ (d).

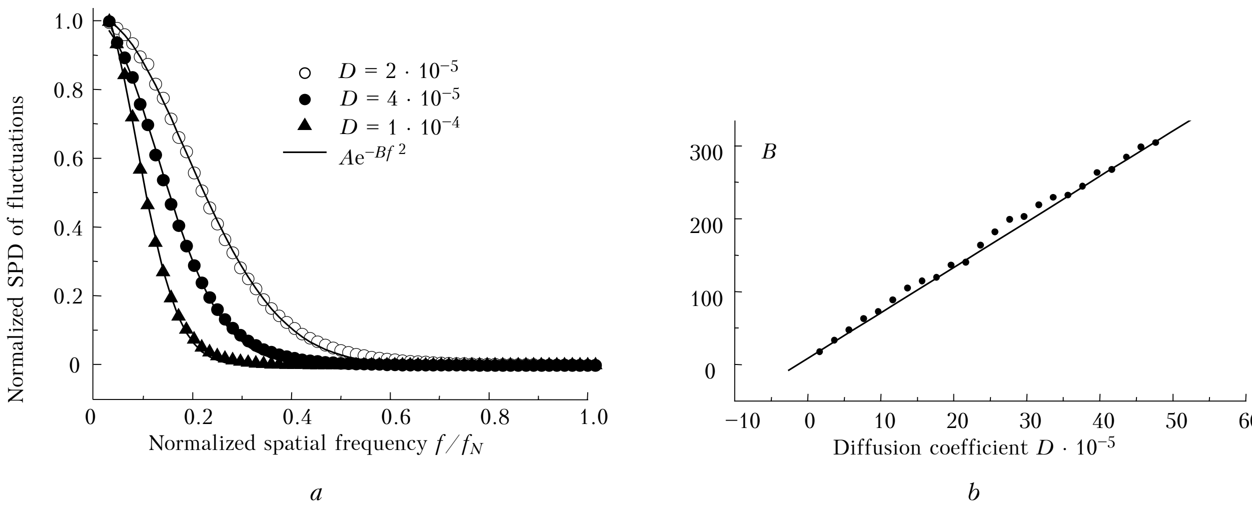

Fig. 3. Spectral characteristics of chaotic modes in the immediate response approximation at $K = 2.65$, $Z_0 = 0$, $\gamma = 1$, $\psi_0 = 0$: averaged SPD of spatial fluctuations (a) and scale relations: dependence of the approximation parameter B on the diffusion coefficient D (b).

The results of numerical studies of the system with the allowance for relaxation (Fig. 4) roughly correspond to the results obtained in the immediate response approximation. Chaotic modes are characterized by oscillating temporal autocorrelation functions of phase fluctuations (Fig. 4*a*) with the oscillation period $T_{osc} \approx 2T$. The SPD of fluctuations also has the decreasing character, but it is better approximated by the functions like $G_{app}(f) = A/(1 + Bf^2)^2$ (Fig. 4*b*).

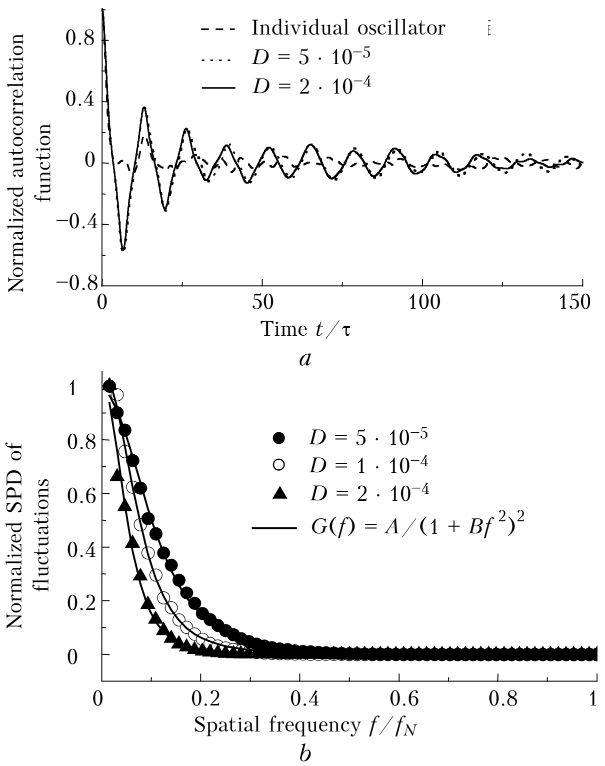

Fig. 4. Statistics of phase fluctuations ($K = 3.65$, $T/\tau = 6$, $Z_0 = 0$, $\gamma = 1$, $\psi_0 = 0$): temporal correlation functions averaged over several spatial points (*a*), SPD (*b*).

Consider the effect of diffraction on the system dynamics. In accordance with the above assumptions, we have found two different types of chaotic modes in the system. These two modes differ by the shape of the SPD of fluctuations: monotonically decreasing for the first mode and oscillating for the second mode (Fig. 5). The transition from the first mode to the second one at a fixed value of the diffusion coefficient occurs as the parameter Z_0 increases, that is, the contribution of diffraction to the system dynamics increases. Figure 6*a* depicts a separatrix of these modes in the region of small values of the parameters on the plane $D - Z_0$; the zone *1* corresponds to the first mode, and the zone *2* – to the second one. It can be easily seen that the separatrix is linear. This can be easily explained using Eq. (2) with the allowance for small value of the parameter Z_0. Actually, because of small exponent in this case, the field amplitude A_{FB} depends linearly on Z_0, and, consequently, both the diffusion and diffraction coefficients are present in the main equation (1) in the same (linear) form, and just that explains the linear character of the separatrix.

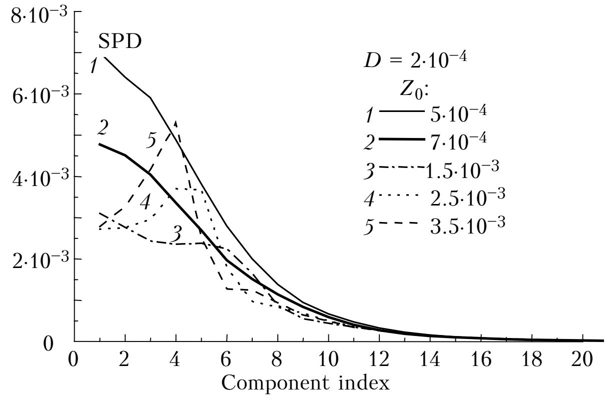

Fig. 5. SPD of phase fluctuations at different values of the diffraction parameter ($K = 2.75$).

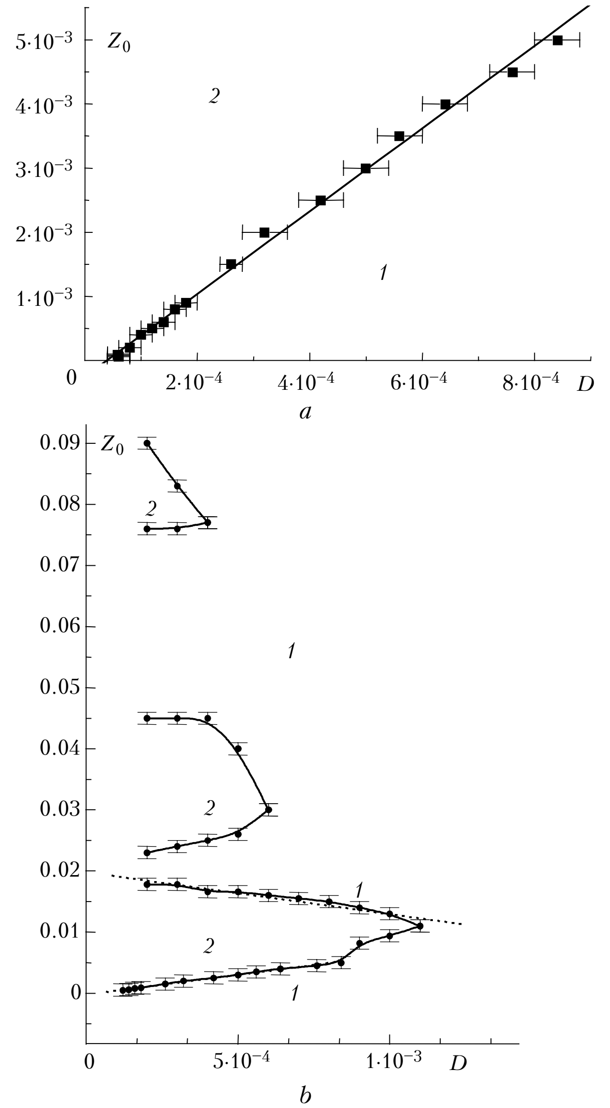

Fig. 6. Separatrix of chaotic modes in the plane of diffraction–diffusion parameters ($K = 2.75$) in the region of small values of the diffraction and diffusion parameters (*a*) and in a wider range of parameter variability (*b*).

For large values of the diffraction coefficient, the dependence $A_{FB}(Z_0)$ has rather nontrivial form, and points separating the modes form singular areas in the plane $D - Z_0$ (Fig. 6*b*). The SPD is nonmonotonic inside these areas and monotonic beyond them. It can be easily seen that at some fixed small value of the diffusion parameter these areas alternate successively at the increase of the diffraction parameter Z_0. Such a behavior is determined by the dependence $A_{FB}(Z_0)$ at large values of Z_0.

Conclusion

As was already mentioned in the Introduction, the interest to studies of statistical properties of chaotic modes in distributed systems with a feedback is caused by the potential possibility of applying such systems to laboratory modeling of optical turbulence. Analysis showed that a delayed system may have modes, in which modulation of the refractive index of the LC transparency is chaotic in space and time, and the SPD of spatial fluctuations has a monotonic decreasing character. It should be noted that the LC transparency in this system can be considered as a phase screen for the reflected light, and the character of fluctuations of the refractive index of this screen corresponds to the common properties of fluctuations of the refractive index of a thin layer of randomly inhomogeneous medium.

It is interesting to follow some analogy between the optical modeling of phase screens used here and numerical simulation of phase screens that is widely used in computer physics.[16] Remind that computer generation of two-dimensional random fields by spectral methods at a 2D spatial grid first assumes specification of some delta-correlated noise. Then the field with the specified profile of the spectrum of spatial fluctuations is generated from this noise through Fourier filtering.

Computer realizations of two-dimensional fields obtained in such a way have, on the one hand, a random character and, on the other hand, needed statistical properties. In the discrete model of a delayed optical system with low diffraction, we have a system of coupled chaotic oscillators, in which the delay leads to formation of random, in space and time, inhomogeneities (analog of a random number generator in the Monte Carlo method), and spatial connections perform additional filtering of fluctuations. In this case, the amplitude Fourier filter in the feedback loop can successfully play the role of a filter in an optical system.

Thus, such an analog mechanism of generation of phase screens reminds of the numerical one. Regardless that this analogy is rather conditional, it well illustrates the advantage of systems with local time instability as potential generators of the controlled optical turbulence in comparison with the known systems, in which the transition to chaotic modes occurs through stochastization of regular structures.

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