Formation of the interferograms in diffusely scattered light in recording Gabor hologram of the focused image of an amplitude screen

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Received January 30, 2001

Recording and reconstruction of a Gabor hologram of the focused image of an amplitude screen to control the axial wave aberration of a lens are analyzed. It is shown theoretically and experimentally that a control error arises in the third order of approximation because of a spherical aberration of the hologram.

In Refs. 1 and 2 it was shown that single-exposure holography of the focused image of an amplitude screen leads, at the stage of hologram reconstruction, to formation of an equal-thickness fringe interferogram in diffusely scattered light. Such an interferogram characterizes spherical aberration of a lens used to form real image of a screen at the stage of the hologram recording. The formation of the real image of the screen in the hologram-recording scheme used in Ref. 3 allows spherical aberration of the controlled lens to be excluded. As a result, at the stage of hologram reconstruction, we record an equal-thickness fringe interferogram that characterizes astigmatism of the controlled lens. Analysis of the interferogram formation in coherent diffusely scattered light in the parabolic approximation ignores possible control errors due to hologram aberrations.

In this paper, peculiarities of the interferogram formation in diffusely scattered light are analyzed in the third-order approximation for the case of recording the Gabor hologram of the focused image of an amplitude screen in order to determine the errors due to axial wave aberrations of a positive lens or an objective.

As shown in Fig. 1, the amplitude screen *1* placed in the plane (x_1, y_1) is illuminated with a coherent radiation of a diverging quasispherical wave having the length of curvature R. With the lens L, whose principal plane is (x_2, y_2), the screen image is formed in the plane (x_3, y_3) of the photographic plate *2*, and the hologram of the focused screen image is recorded at a single exposure.

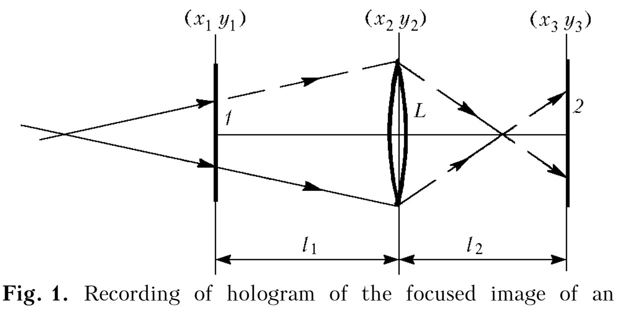


Fig. 1. Recording of hologram of the focused image of an amplitude screen: amplitude screen *1*, photographic plate *2*, controlled lens L.

In the third-order approximation, the distribution of the complex field amplitude in the hologram plane, omitting constant coefficients, has the form

$$u(x_3, y_3) \sim \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} [1 - t(x_1, y_1)] \times$$
$$\times \exp i\left[\frac{k}{2R}(x_1^2 + y_1^2) + \varphi_0(x_1, y_1)\right] \times$$
$$\times \exp -i\left[\frac{k}{8R^3}(x_1^2 + y_1^2)^2\right] \times$$
$$\times \exp i\left\{\frac{k}{2l_1}[(x_1 - x_2)^2 + (y_1 - y_2)^2]\right\} \times$$
$$\times \exp -i\left\{\frac{k}{8l_1^3}[(x_1 - x_2)^2 + (y_1 - y_2)^2]^2\right\} p(x_2, y_2) \times$$
$$\times \exp -i\left[\frac{k}{2f}(x_2^2 + y_2^2) - \varphi(x_2, y_2)\right] \times$$
$$\times \exp i\left\{\frac{k}{2l_2}[(x_2 - x_3)^2 + (y_2 - y_3)^2]\right\} \times$$
$$\times \exp -i\left\{\frac{k}{8l_2^3}[(x_2 - x_3)^2 + (y_2 - y_3)^2]^2\right\} dx_1 dy_1 dx_2 dy_2, \quad (1)$$

where k is the wavenumber; l_1 and l_2 are the separations between the planes (x_1, y_1), (x_2, y_2) and (x_2, y_2), (x_3, y_3), respectively; $t(x_1, y_1)$ is the screen absorption amplitude, which is a random function of coordinates; $\varphi_0(x_1, y_1)$ is a deterministic function, which characterizes, in the general case, phase distortions of the wave used to illuminate the amplitude screen, for example, due to aberrations in the source optical system; $p(x_2, y_2)$ is the pupil function[4] of the lens L with the focal length f; $\varphi(x_2, y_2)$ is a deterministic function, which characterizes axial wave aberrations of the controlled lens.

If the diameter D_0 of the illuminated screen area satisfies the condition $D_0 \geq Rd/(l_1 + R)$, where d is the diameter of the lens L pupil, which restricts the spectrum of spatial frequencies of the screen, then, provided that the equality $1/f = 1/l_1 + 1/l_2$ is

0235-6880/02/07 567-05 $02.00

fulfilled, the distribution of the complex field amplitude in the plane (x_3, y_3) is described by the following equation:

$$u(x_3, y_3) \sim \exp ik\left[\frac{1}{2l_2}(x_3^2+y_3^2)-\frac{1}{8l_2^3}(x_3^2+y_3^2)^2\right]\times$$

$$\times\Big\{[1-t(-\mu x_3, -\mu y_3)]\exp i\,\varphi_0(-\mu x_3, -\mu y_3)\times$$

$$\times\exp i\left[\frac{k\mu^2}{2}\left(\frac{1}{R}+\frac{1}{l_1}\right)(x_3^2+y_3^2)\right]\times$$

$$\times\exp -i\left[\frac{k\mu^4}{8}\left(\frac{1}{R^3}+\frac{1}{l_1^3}\right)(x_3^2+y_3^2)^2\right]A(x_3, y_3)\otimes$$

$$\otimes P(x_3, y_3)\otimes\Phi_1(x_3, y_3)\otimes\Phi_2(x_3, y_3)\Big\}, \qquad (2)$$

where $\otimes$ denotes convolution; $\mu = l_1/l_2$ is the coefficient of the scale transformation;

$$A(x_3, y_3) = \iiiint\limits_{-\infty}^{\infty}\exp i\,\psi_1(x_1, y_1; x_2, y_2)\times$$

$$\times\exp -ik\left[\left(\frac{x_1}{l_1}+\frac{x_3}{l_2}\right)x_2+\left(\frac{y_1}{l_1}+\frac{y_3}{l_2}\right)y_2\right]\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2$$

is a deterministic function caused by aberrations determined by the function

$$\psi_1(x_1, y_1; x_2, y_2) = -\frac{k}{8l_1^3}(6x_1^2x_2^2+6y_1^2y_2^2-4x_1^3x_2-$$

$$-4x_1^2y_1y_2+2x_1^2y_2^2-4x_1x_2^3-4x_1x_2y_1^2+$$

$$+8x_1x_2y_1y_2-4x_1x_2y_2^2+2x_2^2y_1^2-$$

$$-4x_2^2y_1y_2-4y_1^3y_2+2y_1^2y_2^2-4y_1y_2^3);$$

$$P(x_3, y_3) = \iint\limits_{-\infty}^{\infty}p(x_2, y_2)\exp i\,\varphi(x_2, y_2)\times$$

$$\times\exp -i\left[\frac{k}{l_2}(x_2x_3+y_2y_3)\right]\mathrm{d}x_2\,\mathrm{d}y_2$$

is the Fourier transform of the generalized pupil function[5] of the lens L accounting for its axial wave aberrations;

$$\Phi_1(x_3, y_3) = \iint\limits_{-\infty}^{\infty}\exp -i\left[\frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_2^2+y_2^2)^2\right]\times$$

$$\times\exp -i\left[\frac{k}{l_2}(x_2x_3+y_2y_3)\right]\mathrm{d}x_2\,\mathrm{d}y_2$$

is the Fourier transform of the corresponding function;

$$\Phi_2(x_3, y_3) = \iint\limits_{-\infty}^{\infty}\exp i\,\psi_2(x_2, y_2; x_3, y_3)\times$$

$$\times\exp -i\left[\frac{k}{l_2}(x_2x_3+y_2y_3)\right]\mathrm{d}x_2\,\mathrm{d}y_2$$

is the Fourier transform of the function $\exp i\psi_2(x_2, y_2; x_3, y_3)$; the form of the function $\psi_2(x_2, y_2; x_3, y_3)$ is similar to that of $\psi_1(x_1, y_1; x_2, y_2)$ with l_1 replaced by l_2 and the coordinates x_1, y_1 replaced by x_2, y_2, and x_2, y_2 replaced by x_3, y_3.

Since the width of the function $P(x_3, y_3)$ is of the order of $\lambda l_2/d$ (Ref. 6), where λ is the wavelength of radiation used for hologram recording and reconstruction, assume that variation of the phase of a spherical wave with the length of curvature $Rl_2/\mu(R+l_1)$ does not exceed π within the domain of existence of this function. Then in the plane (x_3, y_3) for the diameter $D \le Rd/\mu\,(R+l_1)$ the square phase factor $\exp i\left[\frac{k\mu^2}{2}\left(\frac{1}{R}+\frac{1}{l_1}\right)(x_3^2+y_3^2)\right]$ can be factored out of the convolution integral signs in Eq. (2), keeping in mind the fact that the width of the function $\Phi_1(x_3, y_3)\otimes\Phi_2(x_3, y_3)$ is smaller than that of the function $P(x_3, y_3)$.

Let the photographic layer exposed to radiation with the intensity $u(x_3, y_3)\,u^*(x_3, y_3)$ be processed and a negative image be obtained within the linear part of the characteristic blackening curve. Then for $t(x_1, y_1) \ll 1$ (Ref. 7) the complex transmission amplitude $\tau(x_3, y_3)$ of the hologram neglecting the constant component, because it occupies a small area in the interferogram plane, for the diffusely scattered light is described by the following equation:

$$\tau(x_3, y_3) \sim \Big\{\exp -i\,\varphi_0(-\mu x_3, -\mu y_3)\times$$

$$\times\exp i\left[\frac{k\mu^4}{8}\left(\frac{1}{R^3}+\frac{1}{l_1^3}\right)(x_3^2+y_3^2)^2\right]A^*(x_3, y_3)\otimes$$

$$\otimes P^*(x_3, y_3)\otimes\Phi_1^*(x_3, y_3)\otimes\Phi_2^*(x_3, y_3)\Big\}\times$$

$$\times\Big\{t(-\mu x_3, -\mu y_3)\exp i\,\varphi_0(-\mu x_3, -\mu y_3)\times$$

$$\times\exp -i\left[\frac{k\mu^4}{8}\left(\frac{1}{R^3}+\frac{1}{l_1^3}\right)(x_3^2+y_3^2)^2\right]A(x_3, y_3)\otimes$$

$$\otimes P(x_3, y_3)\otimes\Phi_1(x_3, y_3)\otimes\Phi_2(x_3, y_3)\Big\} + \text{c.c.}, \qquad (3)$$

c.c. means complex conjugate.

The first term in Eq. (3) determines the following diffraction of waves in the (−1)st order, and the complex conjugate term – in the (+1)st order.

Since for the reference wave we can suppose that

$$P(x_3, y_3) \mathrm{E}\ \Phi_1(x_3, y_3) \mathrm{E}\ \Phi_2(x_3, y_3) \mathrm{E}\ \delta(x_3, y_3),$$

where $\delta(x_3, y_3)$ is the Dirac delta, and for the object wave we can suppose that the functions

$\exp i\varphi_0(-\mu x_3, -\mu y_3)$, $\exp i\left[\frac{k\mu^4}{8}\left(\frac{1}{R^3}+\frac{1}{l_1^3}\right)(x_3^2+y_3^2)^2\right]$, and $A(x_3, y_3)$ vary slowly with the coordinate as compared with the function $P(x_3, y_3) \otimes \Phi_1(x_3, y_3) \otimes \otimes \Phi_2(x_3, y_3)$ determining the size of a subjective speckle in the plane (x_3, y_3), the equation for the complex transmission amplitude of the hologram takes the form

$$\tau(x_3, y_3) \sim [t(-\mu x_3, -\mu y_3) \otimes P(x_3, y_3) \otimes \otimes \Phi_1(x_3, y_3) \otimes \Phi_2(x_3, y_3) + t(-\mu x_3, -\mu y_3) \otimes P^*(x_3, y_3) \otimes \otimes \Phi_1^*(x_3, y_3) \otimes \Phi_2^*(x_3, y_3)]. \quad (4)$$

As in Refs. 1–3, at the stage of hologram reconstruction, illumination in the focal plane (x_4, y_4) of the objective installed behind the hologram is recorded. For shorter representation, assume that the objective focal length is equal to l_2. Then, based on Eq. (4), the distribution of the complex field amplitude in the plane (x_4, y_4), neglecting its diffraction limit, is described by the equation:

$$u(x_4, y_4) \sim \Bigg\{\exp i\varphi(-x_4, -y_4) \times \exp -i\left[\frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right] \times \iiiint_{-\infty}^{\infty} \exp i\,\psi_2(x_2, y_2; x_3, y_3) \times \exp -i\left\{\frac{k}{l_2}[(x_2+x_4)\,x_3 + (y_2+y_4)\,y_3]\right\} dx_2 dy_2 dx_3 dy_3 + \exp -i\,\varphi(x_4, y_4) \exp i\left[\frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right] \times \iiiint_{-\infty}^{\infty} \exp -i\,\psi_2(x_2, y_2; x_3, y_3) \times \exp i\left\{\frac{k}{l_2}[(x_2-x_4)\,x_3 + (y_2-y_4)\,y_3]\right\} dx_2 dy_2 dx_3 dy_3\Bigg\} \times p(x_4, y_4)\, F(x_4, y_4), \quad (5)$$

where

$$F(x_4, y_4) = \iint_{-\infty}^{\infty} t(-\mu x_3, -\mu y_3) \times \exp -i\left[\frac{k}{l_2}(x_3 x_4 + y_3 y_4)\right] dx_3\, dy_3$$

is the Fourier transform of the corresponding function.

Let the diffraction field be spatially filtered on the optical axis using an opaque screen with a round aperture at the stage of hologram reconstruction. If the value of the function $\psi_2(x_2, y_2; x_3, y_3)$ within the aperture diameter is smaller than π at least by an order of magnitude, then the distribution of the complex field amplitude in the objective focal plane takes the form

$$u(x_4, y_4) \sim \Bigg\{\exp i\varphi(-x_4, -y_4) \times \exp -i\left[\frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right] + \exp -i\varphi(x_4, y_4) \exp i\left[\frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right]\Bigg\} \times p(x_4, y_4)\, F(x_4, y_4) \otimes P_0(x_4, y_4), \quad (6)$$

where

$$P_0(x_4, y_4) = \iint_{-\infty}^{\infty} p_0(x_3, y_3) \times \exp -i\left[\frac{k}{l_2}(x_3 x_4 + y_3 y_4)\right] dx_3\, dy_3$$

is the Fourier transform of the transmission function $p_0(x_3, y_3)$ of the spatial filter.[4]

If the period of the function

$$\exp i\left[\varphi(-x_4, -y_4) - \frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right] + \exp -i\left[\varphi(x_4, y_4) - \frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right]$$

is at least an order of magnitude[8] larger than the size of the subjective speckle in the plane (x_4, y_4) determined by the width of the function $P_0(x_4, y_4)$, then it can be factored out of the convolution integral sign in Eq. (6). Then, for the even function $\varphi(x_4, y_4)$ the illumination distribution in the objective focal plane is described by the following equation:

$$I(x_4, y_4) \sim \left\{1 + \cos 2\left[\varphi(x_4, y_4) - \frac{k}{8}\left(\frac{1}{l_1^3}+\frac{1}{l_2^3}\right)(x_4^2+y_4^2)^2\right]\right\} \times |F(x_4, y_4) \otimes P_0(x_4, y_4)|^2. \quad (7)$$

It follows from Eq. (7) that the subjective speckle structure is modulated by the interference fringes of equal thickness and, according to Eq. (6), within the pupil image of the controlled lens (Fig. 1). The interference pattern, as in Refs. 1 and 2, characterizes spherical aberration of the lens, if its astigmatism coefficient is assumed to have higher order of smallness as compared with the coefficient of spherical aberration,[4] and if, additionally, the spherical aberration of the hologram is determined by the second term in Eq. (7). If we suppose that deviation from the spherical wave surface across the pupil diameter of the controlled lens does not exceed one tenth of the wavelength, then the diameter should satisfy the following condition

$$d \le 2\sqrt[4]{0.8\lambda\, l_1^3\, l_2^3/(l_1^3 + l_2^3)}$$

neglecting the increasing sensitivity of the interferometer.

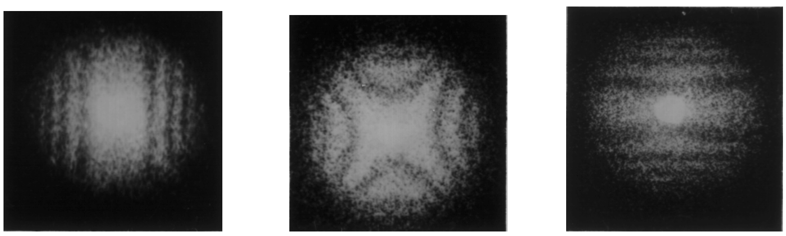
In the experiment, holograms were recorded on Mikrat-VRL photographic plates using He–Ne laser radiation at the wavelength of 0.63 μm. For illustration, the experimental technique consisted in recording of Gabor holograms of the focused image of an amplitude screen for control of astigmatism of a positive lens (or objective)[3] at different values of its pupil diameter. Then the interferograms recorded at the stage of the hologram reconstruction were photometrized, and the resulting experimental data were compared with theoretical predictions. The latter ones followed from the fact that in the third-order approximation in the case that the hologram of the focused image of an amplitude screen is recorded by the method excluding spherical aberration of the controlled lens,[3] the illumination distribution in the objective focal plane at the stage of hologram reconstruction takes the form

$$I(x, y) \sim \left\{1 + \cos 4\left[\varphi(x, y) - \frac{1.125k}{8f^3}(x^2 + y^2)^2\right]\right\} \times \times |F(x, y) \otimes P(x, y)|^2, \quad (8)$$

where designations from Ref. 3 are used.

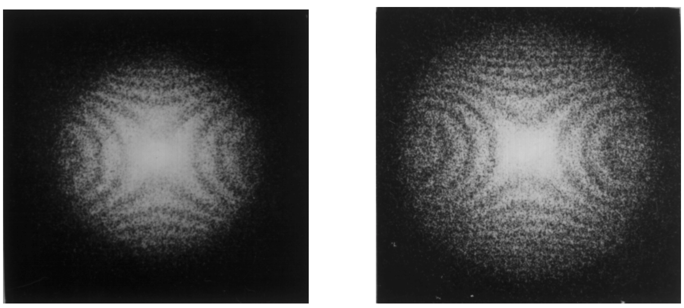
For an example, Fig. 2 depicts interferograms in equal-thickness fringes that characterize astigmatism of the controlled lens with the coefficient $C = 0.5\lambda$, for which the focal length was 130 mm and the pupil diameter was 10 mm. These inteferograms were recorded in the focal plane of the objective with the focal length of 50 mm at spatial filtering of the diffraction field in the hologram plane through hologram reconstruction on the optical axis with a small-aperture (≈ 2 mm) laser beam.

If the interferogram in Fig. 2 characterizes only astigmatism of the controlled lens, then the further increase of the pupil diameter $d \ge 2\sqrt[4]{0.8\lambda f^3}$ leads to the situation that the interferogram in equal-thickness fringes acquires the shape characteristic of combination of the astigmatism and spherical aberration. Thus, Fig. 3 depicts interferograms recorded at the stage of reconstruction of holograms, which were recorded at the pupil diameter of the controlled lens $d_1 = 14$ mm and $d_2 = 18$ mm. The pupil diameter in the experiment was changed by changing the diameter of the diaphragm, which was installed in the plane of formation of the spatial frequency spectrum of the amplitude screen, that is, in the plane of the mirror *4* in Fig. 1 from Ref. 3. Since the values of d_1 and d_2 are within the framework of the approximation considered ($d \le 2\sqrt[6]{1.6\lambda f^5}$), based on Eq. (8) and taking into account that the astigmatism coefficient increases in proportion to the square radius of the lens pupil,[4] we can calculate the positions of minima of the interference fringes on the axes. For the normalized pupil radius of the controlled lens, these positions are shown by dots in Fig. 4 and they correspond to positions of minima of interference fringes on the axes x and y for Fig. 3 accurate to the error of photometry.



a *b* *c*

Fig. 2. Interferograms in equal-thickness fringes recorded in the case of hologram recording at tangential focus (*a*), in the best-focus plane (*b*), and at sagittal focus (*c*).



a *b*

Fig. 3. Interferograms in equal-thickness fringes recorded in the case of hologram recording in the best-focus plane for the pupil diameter of the controlled lens: 14 mm (*a*) and 18 mm (*b*).

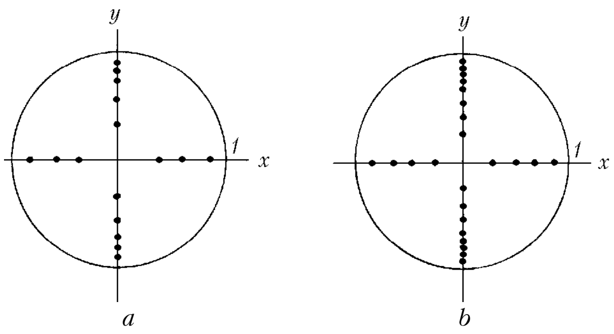


Fig. 4. Positions of minima of interference fringes on the axes x and y for lens pupil diameter: 14 (a) and 18 mm (b).

Thus, the results obtained showed that in recording a Gabor hologram of the focused image of an amplitude screen for control of axial wave aberrations of a positive lens or an objective, a systematic error arises, which should be taken into account in interferogram decoding. This error is caused by spherical aberration of the hologram, when the pupil diameter of the controlled lens or objective does not satisfy the Fresnel diffraction condition.

The study with higher than third order of approximation is difficult, because the frequency of interference fringes increases as the pupil diameter increases (Fig. 3b).

As this takes place, the period of interference fringes becomes comparable with the size of a subjective speckle in the recording plane. As a result, visibility of the interferogram decreases, because interference fringes modulate subjective speckle structure. In this connection, the interferogram contrast can be increased by applying a wide-angle optical system with a rather short focal length (at least an order of magnitude smaller than the focal length of the objective used in the experiment) in order to decrease the speckle size.

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