Algorithm for control over $LiNbO_3$-crystal-based phase-front corrector

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An efficient algorithm for control over a phase-front corrector based on electrooptical crystals has been developed with the use of the method of piecewise linear approximation. The accuracy characteristics of the algorithm are analyzed for the cases assuming the presence of Poisson and Gaussian noise in control channels. The analytical equations are derived for the variance of error and for the correlation coefficients of results of phase front reconstruction, as well as the estimate of computational expenses.

1. Introduction

Turbulence of the medium of propagation significantly deteriorates the performance characteristics of the current optical range finding systems. Phase conjugation systems efficiently compensating for distortions of light beams propagating through perturbed channels are being currently developed.[1,2] Such adaptive optical systems usually use deformable mirrors as the executive optical elements. Although having some advantages, phase-front correctors possess some disadvantages. First, they can hardly give the response function of a preset form. Second, they, being mechanical systems, have such properties as hysteresis and phase delay of a response. One of the promising ways of the development of adaptive optical systems is presentation of the phase front as a sum of spatial modes.

The possibility of using electrooptical $LiNbO_3$ crystals as a phase-front corrector has been studied in Ref. 3. Upon propagation along the z axis through the system of two crystals, one turned by 9°° relative to this axis, the optical radiation acquires the total phase shift

$$\varphi(x, y) = \frac{1}{2} l_0 n_0^3 r_{33} \left\{2 - \frac{E}{l_0} \left[l(x) - l(y)\right]\right\}, \quad (1)$$

where l_0 is the optical radiation path length in the crystal; n_0 is the refractive index of the crystal; E is the electric field strength between the electrodes; r_{33} is the electrooptical coefficient of the crystal; $l(x)$ and $l(y)$ are the functions describing the shape of electrodes in the planes yox and yoz, respectively.

To realize the phase-front corrector capable of compensating for non-stationary distortions of an arbitrary form, one should arrange sequentially N_x correctors producing the correction of the form $l_i(x)$ and N_y correctors producing the correction of the form $l_j(y)$, $i = \overline{1, N_x}$; $j = \overline{1, N_y}$. The sum response of the corrector can be presented as

$$G(x, y) = \sum_{i=1}^{N_x} c_i l_i(x) + \sum_{j=1}^{N_y} d_j l_j(y), \quad (2)$$

where c_i and d_j are the weighting coefficients, which are proportional to the control voltage across the corrector's electrodes.

The shape of the control electrode should change with the crystal width. Therefore, the width of the entrance window of the crystal should far exceed its height. This requirement leads to necessary contraction of the optical beam in the corresponding planes. Thus, one element of the $LiNbO_3$-based phase-front corrector should be a "cylindrical lens − crystal − cylindrical lens" system.

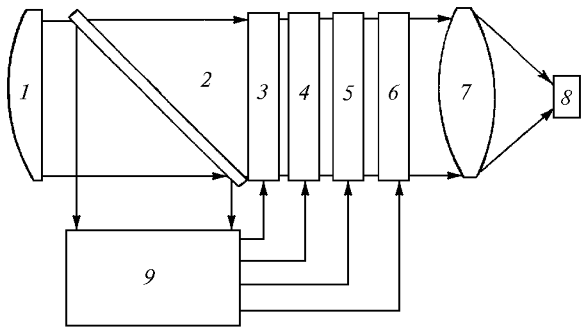

Fig. 1. Structure scheme of an adaptive optical system with the $LiNbO_3$-based phase-front corrector: objective lens (*1*), beam-splitter (*2*), corrector c_1x (*3*), corrector c_2x^2 (*4*), corrector d_1y (*5*), corrector d_2y^2 (*6*), focusing lens (*7*), photodetector (*8*), phase-front sensor (*9*).

Figure 1 shows the structure of a hypothetical adaptive optical system, in which the phase front is corrected with the electrooptical correctors, which allow presentation of the optical wave front as a set of preset spatial modes. To organize control in such a phase conjugation system, one should calculate the parameters proportional, in the general case, to the coefficients c_i and d_j. This problem can be solved based on the method of piecewise linear approximation of the phase front. This method was first described in

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Ref. 4. The efficiency of the obtained algorithm is rather important problem in this case.

Thus, the aim of this paper is to synthesize the algorithm for reconstruction of the phase front in a hypothetical adaptive optical system with the $LiNbO_3$-based phase-front corrector and evaluation of its efficiency.

When estimating the noise error, one should take into account the fact that there is a weak signal in every channel of the Hartmann sensor, because the total intensity of optical radiation incident on one quadrant photodetector of the Hartmann sensor is equal to I/M^2, where M^2 is the number of subapertures. Thus, the signal at the output of each quadrant photodetector must be described by the Poisson distribution density, therefore the efficiency should be analyzed starting from the Poisson model of noise at recording.

2. Calculation of control actions

Let us consider the problem in the following formulation. Phase distortions of the optical radiation are compensated for by the $LiNbO_3$-based phase-front corrector. To calculate the function of control actions, we use the method of piecewise linear approximation of the measurement results on the phase front local tilts.

Let we have a square-shaped aperture consisting of $M\times M$ identical subapertures, in the center of which the phase front local tilts of the form

$$U_{i,j} = k^{-1}\frac{\partial\Psi(x_i\, y_j)}{\partial x}\,; \quad V_{i,j} = k^{-1}\frac{\partial\Psi(x_i\, y_j)}{\partial y} \qquad (3)$$

are measured. In Eq. (3) k is the wave number, and Ψ is the function describing the phase distortion.

Consider the jth ($j = \overline{1, M}$) y cross section of the phase front. In the general case, the cross section of the phase front is a random function of the coordinate x_i ($i = \overline{1, M}$). Divide the cross section of the phase front into M intervals, within each the phase front is approximated by a segment of a straight line:

$$z_i = a_i + U_i x, \qquad (4)$$

where a_i is the phase shift at the ith interval, z_i is the result of piecewise linear approximation of the phase front at this interval. In Eq. (4) and below the subscripts j are omitted.

To find the values of a_i, let us use the method described in Ref. 4. The condition of joining in this case can be written as

$$a_i + U_i x_i = a_{i+1} + U_{i+1} x_i, \qquad (5)$$

where x_i is the value of the coordinate at the point i.

According to Eq. (5), we can write the system of $M{-}1$ linear equations

$$a_{m-1} - a_m = (U_m - U_{m-1})x_{m-1};\ m = \overline{2, M}\,. \qquad (6)$$

One more equation, complement to the system (6), can be obtained from the condition that the phase averaged over the entire aperture should equal zero

$$\sum_{i=1}^{M}\int\limits_{x_{i-1}}^{x_i}(a_i + U_i x)\,\mathrm{d}x = {}^\circ\,. \qquad (7)$$

Taking into account that $x_i = \Delta x\, i$, $\Delta x = L/M$, where L is the size of the sensor aperture, upon integration the Eq. (7) takes the form

$$\sum_{i=1}^{M} a_i = -\frac{\Delta x}{2}\sum_{i=1}^{M} U_i(2i - 1). \qquad (8)$$

Thus, we derive the system of linear equations

$$\left\|\begin{matrix} 1 & -1 & 0 & \ldots & 0 & 0\\ 0 & 1 & -1 & \ldots & 0 & 0\\ . & . & & & . & .\\ 0 & 0 & 0 & \ldots & 1 & -1\\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}\right\| \left\|\begin{matrix} a_1\\ a_2\\ \vdots\\ a_{M-1}\\ a_M \end{matrix}\right\| = \left\|\begin{matrix} b_1\\ b_2\\ \vdots\\ b_{M-1}\\ \gamma \end{matrix}\right\|, \qquad (9)$$

where $b_i = (U_{i+1} - U_i)i\Delta x$; $\gamma = -\dfrac{\Delta x}{2}\sum\limits_{i=1}^{M} U_i(2i - 1)$.

Having solved Eq. (9), we obtain the algorithm for reconstruction of the actual phase front from measured values of its local tilts:

$$a_i = \frac{\Delta x}{M}\left[\sum_{q=i}^{M-1}(M - q)q(U_{q+1} - U_q) - \right.$$
$$\left. - \sum_{q=1}^{i-1} q^2(U_{q+1} - U_q) - \frac{1}{2}\sum_{q=1}^{M}(2q - 1)U_q\right]. \qquad (1^\circ)$$

Applying the procedure (1°) to all rows of the array U and all columns of the array V, we obtain the value of the phase shift at the ijth subaperture along the coordinates x and y, respectively. Let us return to indexing by rows and columns. Then the phase front reconstructed all over the aperture can be written as

$$\Phi_{ij} = \frac{1}{2}(a^x_{i,j} + a^y_{i,j} + U_{i,j}x_i + V_{i,j}y_j), \qquad (11)$$

where $a^x_{i,j}$ and $a^y_{i,j}$ are the results of piecewise linear approximation over rows and columns of the aperture matrix, respectively.

Phase front aberrations are well described by polynomials of no higher than the third order.[1] It follows therefrom that for realization of the phase-front corrector capable of compensating for nonstationary distortions at least by 7°−8°%, it is sufficient to accept in Eq. (2) that $N_x = N_y = 2$, $l_i(x) = x^i$, $l_j(y) = y^i$. The coefficients of the polynomial can be determined simultaneously with estimating the mean tilts. Toward this end, let us use the results of piecewise linear approximation and present the reconstructed phase front in the following form:

$$G(x, y) = c_1 x + d_1 y + c_2 x^2 + d_2 y^2. \qquad (12)$$

The coefficients c_i and d_j can be found using the method of least squares, which has the following form for the problem under consideration:

$$F(G(x,y)) = \sum_{j=1}^{M}\sum_{i=1}^{M}(c_1x + d_1y + c_2x^2 + d_2y^2 - \Phi_{j,i})^2 \rightarrow \min, \tag{13}$$

where $F(G(x, y))$ is the rms deviation.

To determine the coefficients of expansion, let us set the derivatives $\dfrac{\partial F(G(x,y))}{\partial c_h}$; $\dfrac{\partial F(G(x,y))}{\partial d_h}$, $h = 1, 2$, equal to zero. Then we obtain the system of linear equations, which can be presented in the matrix form as

$$\left\| \begin{matrix} \overline{x^2} & \overline{\overline{xy}} & \overline{x^3} & \overline{\overline{y^2x}} \\ \overline{\overline{xy}} & \overline{y^2} & \overline{\overline{x^2y}} & \overline{y^3} \\ \overline{x^3} & \overline{\overline{yx^2}} & \overline{x^4} & \overline{\overline{y^2x^2}} \\ \overline{\overline{xy^2}} & \overline{y^3} & \overline{\overline{x^2y^2}} & \overline{y^4} \end{matrix} \right\| \cdot \left\| \begin{matrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{matrix} \right\| = \left\| \begin{matrix} \Phi x \\ \Phi y \\ \Phi x^2 \\ \Phi y^2 \end{matrix} \right\|, \tag{14}$$

where $\overline{(*)} = \dfrac{1}{M}\sum\limits_{i}^{M}(*)_i$; $\overline{\overline{(*)}} = \dfrac{1}{M^2}\sum\limits_{j}^{M}\sum\limits_{i}^{M}(*)_{j,i}$.

Solution of Eq. (14) by the method of inversion gives the vector of control actions $\| c_1 d_1 c_2 d_2 \|^{\mathrm{T}}$.

3. Evaluation of accuracy and computer power needed

Let us analyze now the noise errors of phase front reconstruction with the use of the method proposed. We consider random parameters, what simplifies the mathematics. The following conditions are taken into account in analysis. The Hartmann-type phase-front sensor of the adaptive optical system records a weak optical signal. The measurement errors at neighbor subapertures of an adaptive optical system are independent. Physically, this can be justified by the fact that optical signals in different channels of the phase-front sensor are recorded with different quadrant photodetectors. At large M the size of the quadrant photodetector is far less than the size of the receiving aperture. Then the signal at the output of each photodetector of the Hartmann sensor is a mixture of a signal and noise. The distribution density of this mixture is described by the following equation[5]:

$$P(n_{i,j}) = \exp[-(\lambda + \mu)] \times \sum_{s=-\infty}^{\infty} I_s(\lambda + \mu) J_{s-n_{i,j}}(\lambda - \mu), \tag{15}$$

where λ and μ are the parameters of Poisson noise at the opposite cells of the quadrant photodetector, I_s is the modified Bessel function of the sth order; $J_{s-n_{i,j}}$ is the Bessel function of the $(s - n_{i,j})$th order.

The following equality:

$$\alpha_{11} = {}^\circ \tag{16}$$

is valid for the correlation moment of the measurement errors at the neighboring subapertures.

The first and second initial moments of the measurement errors in local tilts α_1 and α_2 within the quadrant photodetector can be thought constant and equal to[5]:

$$\alpha_{1\mathrm{P}} = \lambda - \mu; \quad \alpha_{2\mathrm{P}} = \lambda + \mu + \lambda^2 + \mu^2 - 2\lambda\mu;$$
$$\alpha_{1\mathrm{G}} = m_{\mathrm{n}} = {}^\circ; \quad \alpha_{2\mathrm{G}} = \sigma_{\mathrm{n}}^2, \tag{17}$$

for Poisson and Gaussian noise, respectively. In Eq. (17) m_{n} and σ_{n}^2 are the mathematical expectation and variance of the Gaussian noise.

Taking into account the principle of superposition and in view of the linear character of the obtained method for reconstruction, when analyzing the reconstruction errors, signals at the output of the device employing this algorithm can be thought to have the form

$$U_{j,i} = n^x_{j,i}; \; V_{j,i} = n^y_{j,i}, \tag{18}$$

where $n^x_{j,i}$ and $n^y_{j,i}$ are the statistically independent noises of measurements along the rows and columns of the matrix, respectively.

Consider the y cross section of the phase front. The first-order moment of the noise error can be written as follows:

$$\langle n_{i,j}\rangle = \left\langle \frac{\Delta x}{M}\left[\sum_{q=i}^{M-1}(M - q)\, q(n_{i+1,j} - n_{i,j}) - \right.\right.$$
$$\left.\left. - \sum_{q=i}^{i-1} q^2(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i,j}(2q - 1)\right]\right\rangle. \tag{19}$$

In Eq. (19) and below the superscripts are omitted. Upon making simple transformations and neglecting the terms of the order of $1/M^2$ and smaller, we can write Eq. (19) in the following form:

$$\langle n_{i,j}\rangle \approx \left\langle \left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{i+1,j} - n_{i,j}) - \right.$$
$$\left. - \frac{1}{2}\, n_{i,j} \right\rangle \approx -\frac{1}{2}\,\alpha_1. \tag{2°}$$

Thus, with the allowance for Eq. (17), the mathematical expectation of the noise error of reconstruction is equal to:

for Poisson noise

$$m_{\mathrm{P}} = -\frac{1}{2}\,\alpha_{1\mathrm{P}} = -\frac{1}{2}(\lambda - \mu) \;\text{ at } (\lambda - \mu) \neq {}^\circ;$$
$$m_{\mathrm{P}} = {}^\circ \;\text{ at } (\lambda - \mu) = {}^\circ; \tag{21}$$

and for a Gaussian noise model

$$m_{\mathrm{G}} = -\frac{1}{2}\,\alpha_{1\mathrm{G}} = {}^\circ. \tag{22}$$

The second-order moment of the noise error can be presented as

$$\langle n^2_{i,j}\rangle = \left\langle \frac{\Delta x}{M}\left[\sum_{q=i}^{M-1}(M - q)q(n_{i+1,j} - n_{i,j}) - \right.\right.$$

$$- \sum_{q=i}^{i-1} q^2(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i,j}(2q-1) \Bigg] > \times$$

$$\times < \frac{\Delta x}{M}\Bigg[\sum_{q=i}^{M-1} (M-q)q(n_{i+1,j} - n_{i,j}) -$$

$$- \sum_{q=i}^{i-1} q^2(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i,j}(2q-1) \Bigg] > . \quad (23)$$

Taking into account the terms making the largest contributions, we have

$$<n_{i,j}^2> \approx < \left[\left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\, n_{i,j}\right]^2 > \approx$$

$$\approx \left(\frac{M^2}{18} - \frac{M}{18} - \frac{2}{9M}\right)\alpha_2. \quad (24)$$

Thus, the variance of the noise error can finally be presented as

$$\sigma^2 = \left(\frac{M^2}{18} - \frac{M}{18} - \frac{2}{9M}\right)\alpha_2 + \frac{1}{4}\,\alpha_1^2. \quad (25)$$

Taking into account Eq. (17), we derive the equation for estimation of the noise error under the conditions of Poisson noise:

$$\sigma_P^2 = \left(\frac{M^2}{18} - \frac{M}{18} - \frac{2}{9M}\right)(\lambda + \mu - 2\lambda\mu + \lambda^2 + \mu^2) + \frac{1}{4}(\lambda - \mu)^2. \quad (26)$$

From analysis of Eq. (26) it is seen that the noise error is smallest in the case of a plane phase front and equals to

$$\sigma_P^2 = \left(\frac{M^2}{18} - \frac{M}{18} - \frac{2}{9M}\right) 2\lambda \quad \text{at } \lambda = \mu. \quad (27)$$

For the sake of comparison with the known algorithm for reconstruction of the phase front,[6] let us derive the equation for the noise error under the condition of Gaussian noise. With allowance for Eq. (17) this error can be written as

$$\sigma_G^2 = \left(\frac{M^2}{18} - \frac{M}{18} - \frac{2}{9M}\right)\sigma_n^2. \quad (28)$$

Thus, the obtained analytical equations allow comparative analysis with the known algorithms in the accuracy of phase front reconstruction.

The degree of correlation of the results of phase front reconstruction is of great interest. The correlation moment for the results of approximation of the phase front at two arbitrary subapertures can be written as

$$<n_{i,j}\, n_{w,j}> = < \frac{\Delta x}{M}\Bigg[\sum_{q=i}^{M-1} (M-q)q(n_{i+1,j} - n_{i,j}) -$$

$$- \sum_{q=i}^{i-1} q^2(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i,j}(2q-1) \Bigg] > \times$$

$$\times < \frac{\Delta x}{M}\Bigg[\sum_{q=i}^{M-1} (M-q)q(n_{w+1,j} - n_{w,j}) -$$

$$- \sum_{q=i}^{i-1} q^2(n_{w+1,j} - n_{w,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{w,j}(2q-1) \Bigg] > ;$$

$$w \neq \begin{cases} i+1 \\ i-1 \end{cases}. \quad (29)$$

Taking into account the terms making the largest contribution and the condition (16), we have

$$<n_{i,j}\, n_{w,j}> \approx < \left[\left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\, n_{i,j}\right] \times$$

$$\times \left[\left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{w+1,j} - n_{w,j}) - \frac{1}{2}\, n_{w,j}\right] > =$$

$$= < \left(\frac{M^2}{36} - \frac{M}{18} + \frac{2}{9M}\right)(n_{i+1}n_{w+1} - n_i n_{w+1} - n_{i+1}n_w + n_i n_w) -$$

$$- \left(\frac{M}{12} - \frac{1}{6M} - \frac{1}{12}\right)(n_{w+1}n_i - n_w n_i + n_{i+1}n_w - n_i n_w) -$$

$$- \frac{1}{4}\, n_i n_w > = {}^\circ . \quad (3^\circ)$$

Thus, the results of approximation of the phase front at two arbitrary subapertures do not correlate.

The correlation moment for the results of approximation of the phase front at two neighbor subapertures can be written as

$$<n_{i,j}\, n_{i+1,j}> = < \frac{\Delta x}{M}\Bigg[\sum_{q=i}^{M-1} (M-q)q(n_{i+1,j} - n_{i,j}) -$$

$$- \sum_{q=i}^{i-1} q^2(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i,j}(2q-1) \Bigg] > \times$$

$$\times < \frac{\Delta x}{M}\Bigg[\sum_{q=i}^{M-1} (M-q)q(n_{i+2,j} - n_{i+1,j}) -$$

$$- \sum_{q=i}^{i-1} q^2(n_{i+2,j} - n_{i+1,j}) - \frac{1}{2}\sum_{q=1}^{M} n_{i+1,j}(2q-1) \Bigg] > . \quad (31)$$

After transformations similar to Eq. (3°) and taking into account Eq. (17), we have

$$<n_{i,j}n_{i+1,j}> \approx < \left[\left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{i+1,j} - n_{i,j}) - \frac{1}{2}\, n_{i,j}\right] \times$$

$$\times \left[\left(\frac{M}{6} - \frac{1}{3M} - \frac{1}{6}\right)(n_{i+2,j} - n_{i+1,j}) - \frac{1}{2}\, n_{i+1,j}\right] > =$$

$$<n_{i,j}n_{i+1,j}> \approx \left(\frac{-M^2}{36} + \frac{M}{18} + \frac{2}{9M} + \frac{1}{2}\right)(\lambda + \mu - 2\lambda\mu + \lambda^2 + \mu^2). \quad (32)$$

Thus, the correlation moment of the results of phase front approximation at the neighbor subapertures is proportional to the measurement error and depends on the number of subapertures.

Let us estimate the computer power needed for realization of the proposed method of phase front reconstruction. To determine one value of $a_{i,j}$, a computer must execute $7M^2/2 + 3M^2 + 3$ operations. The number of subapertures is equal to M^2, and two measurements of local tilts are performed at each subaperture. So, the calculation of $a_{i,j}$ all over the aperture requires $2M^2(7M^2/2 + 3M^2 + 3)$ operations. Taking into account the operations needed for joining the measurement results in the planes *zox* and *zoy*, the computer power needed for realization of the whole algorithm can be presented as

$$Q = 13M^4 + 8M^2. \tag{33}$$

Equation (33) allows comparative analysis with the known algorithms.

4. Conclusions

Thus, in this paper we propose the structure of a few-parameter phase-front corrector capable of compensating for nonstationary distortions of an arbitrary form and the algorithm for control over it. Application of the $LiNbO_3$ electrooptical crystals allows us to exclude such phenomena as hysteresis and the phase delay of the response.

The analytical equations are derived for estimation of the noise error and the correlation moments of the results of phase-front reconstruction at the neighbor subapertures against the background of Poisson and Gaussian noise. These equations can be used for analyzing the quality of phase conjugation algorithms. In the case of the plane phase front, the noise error takes its minimum value against the background of Poisson noise. The results of phase front reconstruction at arbitrary subapertures are independent; the correlation properties manifest themselves only between the neighboring subapertures and are proportional to the measurement error of local tilts.

The proposed method has been implemented on Pentium 166MMX PC with the MATHCAD-7 Pro application. The results of simulation and comparative analysis with the algorithm of phase front reconstruction from Ref. 6 have shown that, although our method gives a little bit lower accuracy of phase front reconstruction, it requires about $M(^\circ.^\circ 8M^2 + {}^\circ.5M)$ times less computer operations. The algorithm is versatile and can be implemented with the use of analog devices[7] and modern high-efficient parallel computer systems.

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